Fall 2018 COT 3100 Section 2 Homework #1 Solution

1) Fill out the following truth table:

p	q	r	$(\neg q \vee r)$	$\neg(\neg q \vee r)$	$(p \wedge \neg(\neg q \vee r))$	$p \wedge q$	$(p \wedge \neg(\neg q \vee r)) \vee (p \wedge q)$
F	F	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	T	F	F	T	F	F	F
F	T	T	T	F	F	F	F
T	F	F	T	F	F	F	F
T	F	T	T	F	F	F	F
T	T	F	F	T	T	T	T
T	T	T	T	F	F	T	T

2) Use the laws of logic to prove the two following expressions are logically equivalent:

(a) $p \wedge (\overline{\bar{r} \wedge \bar{p}} \wedge (\bar{p} \rightarrow s))$ (b) p

Solution:

Starting with (a) we will attempt to simplify it so as to arrive at (b).

$p \wedge (\overline{\bar{r} \wedge \bar{p}} \wedge (\bar{p} \rightarrow s))$

$p \wedge ((r \vee p) \wedge (\bar{p} \rightarrow s))$ ⇔ De Morgan's Law, and two applications of Double Negation

$p \wedge ((r \vee p) \wedge (\bar{\bar{p}} \vee s))$ ⇔ Law of Implication

$p \wedge ((r \vee p) \wedge (p \vee s))$ ⇔ Double Negation

$p \wedge ((p \vee r) \wedge (p \vee s))$ ⇔ Commutative Law

$p \wedge (p \vee (r \wedge s))$ ⇔ Distributive Law

p ⇔ Law of Absorption

Note: There are many, many ways to show the equivalence of these two expressions.

3) Use the laws of implication to complete the following argument:

1. $p \vee q$
2. $p \rightarrow s$
3. $q \rightarrow r$
4. $\bar{r}$
5. $s \rightarrow (t \wedge u)$

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t

(Note: Numbers 1 – 5 are the premises and proposition below is what is to be deduced from those premises.)

Solution:

1. $q \rightarrow r \ (premise)$
2. $\bar{r} \qquad (premise)$
3. $\bar{q} \qquad (Modus\ Tollens\ with\ 1\ and\ 2)$
4. $(p \vee q)$ (premise)
5. $p \qquad (disjunctive\ elimination\ using\ 3\ since\ \bar{q}\ \ is\ true\ therefore\ q\ is\ false)$
6. $p \rightarrow s \qquad (premise)$
7. $s \qquad (Modus\ Ponens\ with\ 5\ and\ 6)$
8. $s \rightarrow (t \wedge$ u) (premise)
9. $t \wedge u \qquad (Modus\ Ponens\ with\ 7\ and\ 8)$
10. $t \qquad (Rule\ of\ Conjunctive\ Simplification)$

4) Use the rules of propositional logic and laws of inference to prove the following argument:

1. $\neg((\neg p \vee \neg p) \wedge (\neg p \vee \neg q))$
2. $\neg p$

$\neg r$

(Note: Numbers 1 – 2 are the premises and proposition below is what is to be deduced from those premises.)

Solution:

1. $\neg\big((\neg p \vee \neg p) \wedge (\neg p \vee \neg q)\big)$ Premise
2. $\big(\neg(\neg p \vee \neg p) \vee \neg(\neg p \vee \neg q)\big)$ De Morgan's Law on #1
3. $\big(\neg(\neg p) \vee (\neg\neg p \vee \neg\neg q)\big)$ Idempotent Law, De Morgan's Law on #2
4. $\big(p \vee (p \wedge q)\big)$ Double Negation applied 3 times on #3
5. p Law of Absorption
6. $\neg p$ Premise
7. $(p \ \wedge \neg p)$ Rule of Conjunction with #5 and #6

8. F Inverse Law
9. $F \vee \neg r$ Disjunctive Amplification
10. $\neg r$ Identity Law

The key here is realizing that we can get to the deduction False by using the premises. This means the premises themselves are contradictory. Once we get to false, we can use Disjunctive Amplification to attach False or Anything, and then use Identity Law to derive Anything.

5) Prove or disprove the following statements. Use the domain of real numbers for each variable, unless otherwise stated in the problem.

Solution:

a) $\exists x[x^2 + 4x + 3 > 2x^2 + 5x + 6]$

$\exists x[x^2 + x + 3 < 0]$ (bring all terms to one side of the inequality)

$$\exists x \left[(x)^2 + 2x(1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 3 < 0\right] \ (using\ completing\ the\ square, factorize\ the\ equation)$$

$$\exists x \left[\left(x + \frac{1}{2}\right)^2 + \frac{11}{4} < 0\right]$$

$$\exists x \left[\left(x + \frac{1}{2}\right)^2 < -\frac{11}{4}\right]$$

Since there are no real solutions to the following inequality, we can conclude that the given statement is False.

b) $\forall x \in Z^+[(x+1)^2 > x^2]$

$\forall x \in Z^+[(x^2 + 2x + 1) - x^2 > 0]$

$\forall x \in Z^+[2x + 1 > 0]$

$\forall x \in Z^+ \left[x > -\frac{1}{2}\right]$

The above statement holds true for all positive values of x therefore, (b) is True.

c) $\exists x[\forall y(y^2 - 3xy + x^2 \geq 0)]$

The statement holds true when x = 0.

$\exists x[\forall y(y^2 - 3(0)y + (0)^2 \geq 0)]$

$\exists x[\forall y(y^2 \geq 0)]$

$y^2 \geq 0$ *holds true for all real values of* y therefore, (c) is True.

d) $\exists x[\forall y(y^2 - 3xy + x^2 > 0]$
We can attempt this problem by completing the square.
$\exists x[\forall y(y^2 - 3xy + x^2 > 0]$
$\exists x[\forall y((y)^2 - 2y\frac{3x}{2} + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^2 + x^2 > 0)]$
$\exists x[\forall y(\left(y - \frac{3x}{2}\right)^2 - \frac{5x^2}{4} > 0)]$

The minimum point for the above expression is $\left(\frac{3x}{2}, -\frac{5x^2}{2}\right)$. $-\frac{5x^2}{2} > 0$ is not true for any value of x therefore we conclude that (d) is indeed False.

e) $\exists x \in Z^+[\exists y \in Z^+(x^y \; has \; precisely \; 7 \; divisors)]$
In order to prove this statement, we need a pair of x and y that satisfy the given condition. When x = 64 and y = 1, the divisors are then: 1, 2, 4, 8, 16, 32, 64. Since there is a total of 7 divisors, (e) is True. (Note: any prime number raised to the sixth power will suffice as a valid case to prove the assertion.)

6) A wonderful advance in theoretical computer science was the Cook-Levin Theorem, discovered in the early 1970s. Both Cook and Levin independently discovered that the Satisfiability Problem was NP-Complete. Clearly define and summarize the Satisfiability Problem and give one specific instance of a Boolean expression that is satisfiable and another specific instance of a Boolean expression that isn't satisfiable.

Solution:

In Computer Science, the satisfiability problem attempts to identify whether a Boolean expression evaluates to True for at least one combinations of truth assignments to the variables in the expression. For instance,

Example 1: $p \vee q$

If we attempt to solve this expression using the different states of p and q, we can conclude that this Boolean expression is satisfiable. (when either p or q are true, the expression evaluates to True.)

Example 2: $p \wedge \bar{p}$
Since it is just one proposition, p, that we are considering regardless of the state of p the expression always evaluates to False. (when p is True, T ∧ F => F, when p is False, F ∧ T => F). Therefore, the expression from example 2 is unsatisfiable.

It is important to note that these examples are very simple. The SAT problem is one of the many difficult problems encountered in Computer Science. Since the Boolean expression can go on infinitely, there is no one algorithm capable of determining the satisfiability of said expression.