Outline of Intro to Discrete Exam #2 Topics

I. Number Theory

a. Division Algorithm
b. Euclid's Algorithm
c. Extended Euclid's Algorithm
d. Full Solution to $ax+by = c$ for integers given a,b,c.
e. Finding modular inverses
f. Divisibility proofs
g. Pi notation
h. Fundamental Thm of Algebra
i. Least Common Multiple (LCM)
j. Connection between LCM and GCD
k. Idea behind proofs relating lcm, gcd with 3 integers.
l. Calculating # of divisors of an integer.
m. Calculating the sum of divisors of an integer.
n. Calculating the number of times prime p divides into n!
o. Proof that $\sqrt{2}$ is irrational.
p. if p prime $0<a<p$, then $\{a,2a,\ldots,(p-1)\}$ distinct mod p

II. Arith Geo Series, Sums, Matrices

a. Arithmetic Series - solving for terms, sums, etc.
b. Geometric Series - solving for terms, sums, etc.
c. How to recursively define sequences
d. Definition of summation notation
e. Summation Rules
f. Index shift idea
g. Telescopic sum idea
h. Idea of bounding sums with integrals
i. Matrix Addition, Subtraction, Multiplication

III. Mathematical Induction

- **a. Base Case**
- **b. Inductive Hypothesis**
- **c. Inductive Step**
- **d. Summation Rules**
- **e. Not all induction problems use summations**
- **f. How to deal with inequalities**
- **g. Strong Induction**
- **h. Divisibility Problems**
- **i. Matrix Exponentiation Problems**
- **j. Problems with recursively defined sequences**
- **k. Problems with Harmonic numbers**
- **l. Unorthodox Examples - NIM, Nuggets, Trominos,**

Reading from Textbook

Number Theory: 4.1 - 4.4
Sums, Matrices: 2.4, 2.6
Mathematical Induction: 5.1 - 5.3

What to Study

1) Read the notes.

2) Skim textbook.

3) Practice problems posted online in my archive.

4) Look over written lectures and past homework solutions.

Sample Questions

1) Use induction to prove that $64 \mid (3^{2n} - 8n - 1)$ for all integers $n \geq 0$.

2) Let c be an integer such that $3 \mid c$. Prove that $(c+1)^3 \equiv 1 \pmod 9$.

3) Prove the following inequality for all positive integers n:

$$\sum_{i=1}^{2^n} \log_2 i \leq (n-1)2^n + 1$$

(Hint : Remember that $\log_2(2^x - y) \leq x$ when x is a positive integer and y is a non-negative integer such that $y < 2^x$.)

4) Let a and b be integers such that $7 \mid (2a + 3b)$. Prove that $7 \mid (13a + 16b)$.

5) Determine 59^{-1} mod 203. Please express your answer as an integer in between 0 and 202. In order to earn full credit you must use the Extended Euclidean Algorithm.

6) Let $a = 2^3 3^5 5^2 7^1$ and $b = 3^3 5^6 7^2$. How many divisors does a have? How many divisors does b have? Express the greatest common divisor of a and b in prime factorized form. Express the least common multiple of a and b in prime factorized form.

7) Using induction on n, prove for all non-negative integers n that $9 \mid (2^{2n} + 6n - 1)$.

8) Using induction on n, prove for all positive integers n that $\sum_{i=1}^{n} i^2 \leq n^3$.

9) Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}$$

10) Define the sequence t_n as follows: $t_0 = 7$, $t_1 = 10$, $t_n = 3t_{n-1} - 2t_{n-2}$, for all integers $n \geq 2$. Prove, using strong induction on n with 2 base cases, that for all non-negative integers n,

$$t_n = 4 + 3(2^n).$$