Fall 2018 COT 3100 Exam #2 (10/23/2018) (Note: Out of 100 points) - Pages 1, 2 Solutions

1) (8 pts) What is the result of the following matrix computation?

$$\begin{bmatrix} 9 & 6 & 3 \\ 7 & 4 & 7 \end{bmatrix} \times \begin{bmatrix} 10 & 2 \\ 5 & 7 \\ 10 & 3 \end{bmatrix} = \begin{bmatrix} 9(10)+6(5)+3(10) & 9(2)+6(7)+3(3) \\ 7(10)+4(5)+7(10) & 7(2)+4(7)+7(3) \end{bmatrix}$$

$$= \begin{bmatrix} 90+30+30 & 18+42+9 \\ 70+20+70 & 14+28+21 \end{bmatrix} = \begin{bmatrix} 150 & 69 \\ 160 & 63 \end{bmatrix}$$

Grading: 2 pts per entry, 1 pt if correct sum of products written but value obtained is incorrect.

2) (10 pts) What is a closed-form expression in terms of the positive integer *n* for the summation below:

$$\sum_{i=n}^{2n} (\sum_{j=1}^{2i} ij\ )$$

$$\sum_{i=n}^{2n}\left(\sum_{j=1}^{2i} ij\right) = \sum_{i=n}^{2n}(i\left(\sum_{j=1}^{2i} j\right))$$

$$= \sum_{i=n}^{2n}(i\left(\frac{2i(2i+1)}{2}\right))$$

$$= \sum_{i=n}^{2n}(i^2(2i+1))$$

$$= \sum_{i=n}^{2n}(2i^3+i^2)$$

$$= \sum_{i=1}^{2n}(2i^3+i^2) - \sum_{i=1}^{n-1}(2i^3+i^2)$$

$$\frac{2(2n)^2(2n+1)^2}{4} + \frac{(2n)(2n+1)(4n+1)}{6} - \frac{2(n-1)^2n^2}{4} - \frac{(n-1)n(2n-1)}{6}$$

Grading: 1 pt factor out i, 3 pts sum j, 1 pt split 2i^3 + i^2, 1 pt split sum, 1 pt plug in each formula

3) (14 pts) Prove using induction on n that for all positive integers n, $11 \mid (13^n - 2^n)$.

Base case: $n = 1$, $13^1 – 2^1 = 11$, since $11 \mid 11$, the assertion holds for $n = 1$. **(1 pt)**

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that $11 \mid (13^k – 2^k)$. Namely, there exists an integer c such that $13^k – 2^k = 11c$. **(2 pts)**

Inductive step: Prove for $n = k+1$ that $11 \mid (13^{k+1} – 2^{k+1})$. Thus, we must show there exists some integer d such that $13^{k+1} – 2^{k+1} = 11d$. **(2 pts)**

$13^{k+1} – 2^{k+1} = 13(13^k) – 2(2^k)$ **2 pts**
$= (11 + 2)(13^k) – 2(2^k)$ **2 pts**
$= 11(13^k) + 2(13^k) – 2(2^k)$ **1 pt**
$= 11(13^k) + 2(13^k – 2^k)$ **1 pt**
$= 11(13^k) + 2(11c)$, using the inductive hypothesis. **2 pts**
$= 11(13^k + 2c)$ **1 pt**

Since k is a positive integer and c is an integer, it follows that $d = 13^k + 2c$ is also an integer. Thus, we've shown that $11 \mid 13^{k+1} – 2^{k+1}$, proving the inductive step. It follows that the given assertion is true for all positive integers n.

Fall 2018 COT 3100 Exam #2 (10/23/2018) (Note: Out of 100 points) - Pages 3, 4 Solution

4) (15 pts) Find all integer solutions to the equation $177x + 78y = 18$.

Note that 3 divides evenly into 177, 78 and 18, leaving us with the equivalent equation:

$59x + 26y = 6$

If one doesn't realize this, the Euclidean Algorithm would reveal it. Now our goal is to find all solutions to this equivalent equation. Run the Extended Euclidean Algorithm with 59 and 26:

59 = 2 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1 **(4 pts)**

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
5 – 2 x 7 + 2 x 5 = 1
3 x 5 – 2 x 7 = 1
3(26 – 3 x 7) – 2 x 7 = 1
3 x 26 – 9 x 7 – 2 x 7 = 1
3 x 26 – 11 x 7 = 1
3 x 26 – 11(59 – 2 x 26) = 1
3 x 26 – 11 x 59 + 22 x 26 = 1
25 x 26 – 11 x 59 = 1 **(7 pts)**

Now, multiply this equation through by 6 to yield:

6 x 25 x 26 – 6 x 11 x 59 = 6 x 1
150 x 26 – 66 x 59 = 6 **(2 pts)**

Thus, one solution to the given equation is x = -66, y = 150. It follows that all solutions are of the form: $\{(x, y) \mid x = -66 + 26n,\ y = 150 - 59n,\ \forall n \in Z\}$. Plugging in n = 2 yields and rewriting the set yields the equivalent answer $\{(x, y) \mid x = -14 + 26n,\ y = 32 - 59n,\ \forall n \in Z\}$. **(2 pts for all sols)**

5) (15 pts) Find the ***sum of the divisors*** of 225,000, leaving your answer in ***prime factorized form.*** (Hint: First prime factorize the given integer. Use the formula from class to express the sum of divisors as a product of some fractions. It may be helpful for you to use the following factoring formula: $x^3 - 1 = (x - 1)(x^2 + x + 1)$. Then, cancel as necessary and express what remains in prime factorized form. This will require a bit of hand calculation, but nothing that you calculate by hand should exceed 1,000. It also may help you to know that $5^4 = 625$.)

First, prime factorize 225,000:

$225000 = 225 \text{ x } 1000 = 15 \text{ x } 15 \text{ x } 10^3 = 3 \text{ x } 5 \text{ x } 3 \text{ x } 5 \text{ x } 2^3 \text{ x } 5^3 = 2^3 \text{ x } 3^2 \text{ x } 5^5$.

Now, apply the formula for the sum of divisors to get, in factorized form:

$$\frac{(2^4-1)}{(2-1)} \times \frac{(3^3-1)}{(3-1)} \times \frac{(5^6-1)}{(5-1)}$$

Simplify the first two terms exactly (since they are small) and factorize the third term using the hint with $x = 5^2$:

$$15 \times 13 \times \frac{(5^2-1)(5^4+5^2+1)}{(5-1)}$$

Cancel the first term in the fraction with the denominator and use the rest of the hint to simplify the second term in the fraction:

$$15 \times 13 \times 6 \times (625+25+1)$$

$$3 \times 5 \times 13 \times 2 \times 3 \times 651$$

Now, we're at the home stretch, use trial division into 651:

$$3 \times 5 \times 13 \times 2 \times 3 \times 3 \times 217$$

$$3 \times 5 \times 13 \times 2 \times 3 \times 3 \times 7 \times 31$$

Finally, coalesce terms to get:

$$\mathbf{2^1 3^3 5^1 7^1 13^1 31^1}$$

Grading: 4 pts prime factorization, 4 pts plugging into sum of divisors formula, 2 pts getting 15 and 13, 3 pts for factorizing $5^6 - 1$ using hint, 2 pts for finishing up

Fall 2018 COT 3100 Exam #2 (10/23/2018) (Note: Out of 100 points) - Pages 5, 6 Solution

6) (10 pts) Let t_n be defined as follows: $t_0 = -1$, $t_1 = 2$, $t_n = 5t_{n-1} - 6t_{n-2}$, for all integers $n \ge 2$. Prove, using strong induction on n, that for all non-negative integers n, $t_n = 4(3^n) - 5(2^n)$.

Base cases: $n = 0$, LHS $= t_0 = -1$, RHS $= 4(3^0) – 5(2^0) = 4 – 5 = -1$
$n = 1$, LHS $= t_1 = 2$, RHS $= 4(3^1) - 5(2^1) = 12 – 10 = 2$
Thus the given statement is true for both n = 0 and n = 1.

Inductive hypothesis: Assume for all non-negative integers $n \le k$, where k is an arbitrary positive integer, that $t_n = 4(3^n) - 5(2^n)$.

Inductive step: Prove for n = k+1 that $t_{k+1} = 4(3^{k+1}) - 5(2^{k+1})$.

$t_{k+1} = 5t_k – 6t_{k-1}$
$= 5(4(3^k) - 5(2^k)) – 6(4(3^{k-1}) - 5(2^{k-1}))$, using the IH for n=k and n-k-1, which is valid, Since $k \ge 1$, so $k – 1 \ge 0$, and we know that the IH holds for all non-negative integers.
$= 20(3^k) – 25(2^k) – 24(3^{k-1}) + 30(2^{k-1})$
$= 20(3^k) – 25(2^k) – 8(3^k) + 15(2^k)$
$= 12(3^k) – 10(2^k)$
$= 4(3^{k+1}) – 5(2^{k+1})$, proving the inductive step.

It follows that the given statement is true for all non-negative integers n, as desired.

Grading: base cases – 2 pts total
IH – 1 pt
IS – 1 pt
Plug in recurrence – 1 pt
Plug in both IHs – 2 pts
Algebra – 3 pts

7) (10 pts) Let a be a positive real number with a ≥ 2. Using induction on n, prove for all non-negative integers n that

$$\sum_{i=0}^{n} a^i < a^{n+1}$$

Base case: n = 0, LHS = $\sum_{i=0}^{0} a^i = 1$, RHS = $a^{0+1} = a$, since $a \geq 2$, it follows that RHS > LHS for n = 0 and the base case holds.

Inductive Hypothesis: Assume for an arbitrary non-negative integer n = k that

$$\sum_{i=0}^{k} a^i < a^{k+1}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=0}^{k+1} a^i < a^{k+2}$$

$$\sum_{i=0}^{k+1} a^i = \left(\sum_{i=0}^{k} a^i\right) + a^{k+1}$$

$$< a^{k+1} + a^{k+1}, \text{ using IH}$$

$$= 2a^{k+1}$$

$$\leq a(a^{k+1}), \text{ since a} \geq 2.$$

$$= a^{k+2}$$

Thus, we've proven the inductive step. It follows that the given assertion is true for all non-negative integers n.

Grading: base case – 2 pts, IH – 1 pt, IS – 1 pt, 2 pts split sum, 2 pts plug in IH, 1 pt $2a^{k+1}$, 1 pt sub a.

Fall 2018 COT 3100 Exam #2 (10/23/2018) (Note: Out of 100 points) - Page 7 Solution

8) (8 pts) Let a_1, a_2, a_3, … form an arithmetic sequence with $a_{10} = 13$ and $a_{30} = 53$. Determine the sum of the first 20 terms of the sequence, $\sum_{i=1}^{20} a_i$. Put a box around your final answer.

$53 – 13 = a_{30} – a_{10} = (a_1 + 29d) – (a_1 + 9d) = 20d$

It follows that d = 2.

$a_{10} = a_1 + 9d$
$13 = a_1 + 18$
$a_1 = -5$
$a_{20} = a_1 + 19d = -5 + 19(2) = 33$

$S_{20} = \frac{(a_1+a_{20})20}{2} = 10(-5+33) = \mathbf{280}$

Grading: 3 pts to get d, 2 pts to solve for a_1, 2 pts to solve for a_{20}, 1 pt final answer

9) (8 pts) Prove for all positive integers a, b and c: if a = gcd(b, c), then a^2 | (bc).

Since a | b and a | c, it follows that there exist integers d and e such that b = da and c = ea.

bc = (da)(ea) = (de)a^2, since d and e are integers de is an integer and it follows that a^2 | bc as desired.

Grading: 2 pts for rewriting b and a times int, 2 pts for rewriting c and a times different int, 2 pts for plugging into bc, 2 pts for conclusion.

10) (2 pts) Which office supply company bought the naming rights to the Staples Center in LA?

Staples