Fall 2018 COT 3100 Exam #1 (9/14/2018) Solutions

1) (8 pts) Complete filling out the truth table below that evaluates the logical expression. For ease of reading, **please use 0 for false and 1 for true.**

$$(p \lor \bar{q}) \rightarrow (\bar{r} \rightarrow \bar{q}).$$

p	q	r	$p \lor \bar{q}$	$\bar{r} \rightarrow \bar{q}$	$(p \lor \bar{q}) \rightarrow (\bar{r} \rightarrow \bar{q})$
F	F	F	1	1	1
F	F	T	1	1	1
F	T	F	0	0	1
F	T	T	0	1	1
T	F	F	1	1	1
T	F	T	1	1	1
T	T	F	1	0	0
T	T	T	1	1	1

Grading: 1 pt for each row, to earn the point, all three entries on the row have to be correct.

2) (8 pts) Using the laws of logic, show that two following expressions are logically equivalent.

$$(p \land (p \lor \bar{r} \lor q)) \lor ((q \land r) \lor (q \land \bar{r})) \qquad \qquad \bar{p} \rightarrow q$$

Note: You may not use all of the rows shown below.

Step	Reason
1. $(p \land (p \lor \bar{r} \lor q)) \lor ((q \land r) \lor (q \land \bar{r}))$	Given
2. $(p \land (p \lor \bar{r} \lor q)) \lor (q \land (r \lor \bar{r}))$	Distributive Law
3. $(p \land (p \lor \bar{r} \lor q)) \lor (q \land T)$	Inverse Law
4. $(p \land (p \lor \bar{r} \lor q)) \lor (q)$	Identity Law
5. $(p \land (p \lor (\bar{r} \lor q))) \lor (q)$	Associative Law
6. $(p) \lor (q)$	Absorption Law
7. $(\bar{\bar{p}}) \lor (q)$	Double Negation Law
8. $\bar{p} \rightarrow q$	Implication Identity
9.	
10.	
11.	
12.	

Grading: 8 pts for any correct response, 3 pts off if all reasons missing, 1 pt off per error (cap at 8 off). <u>No points</u> off for skipping Associative, Commutative or Double Negation.

3) (8 pts) Use the rules of inference to make the following argument:

$s \rightarrow (p \rightarrow r)$
$s \vee t$
$\bar{t}$
$p \vee q$

$q \vee r$

Step	Reason
1. $s \vee t$	Premise
2. $\bar{t}$	Premise
3. s	Disjunctive Syllogism with #1, #2
4. $s \rightarrow (p \rightarrow r)$	Premise
5. $p \rightarrow r$	Modus Ponens with #3, #4
6. $\bar{p} \vee r$	Implication Identity
7. $p \vee q$	Premise
8. $q \vee r$	Rule of Resolution
9.	
10.	
11.	
12.	

Grading: 8 pts for any correct response, 3 pts off if all reasons missing, 1 pt off per error (cap at 8 off). If none of the premises are explicitly listed in the area where I've asked them to be 4 points off.

4) (5 pts) For an open statement P(x, y), it is known that $\exists x[\forall y(P(x,y)]$. Is it necessarily true that $\forall y[\exists x(P(x,y)]$? If it is the case, prove it, if the assertion is false, create a specific open statement P(x, y) for which $\exists x[\forall y(P(x,y)]$ is true and $\forall y[\exists x(P(x,y)]$ is false and explain why the first is true and the second is false.

In the given information, we are told that there exists some value x for which, for all y, P(x, y) is true. Let this particular value of x be x = a. Thus, for all values of y, P(a, y) is true.

Now, let's look at attempting to make a statement for all values y. We must determine, if, for all values of y, if there exists some value of x, for which P(x, y) is true. Thus, if b is an arbitrarily chosen element taken from the universe of values that y can be chosen from, we must find at least one value of x for which P(x, b) is true. Luckily, no matter which b is chosen, we know for a fact that P(a, b) is true. Thus, it DOES follow that $\exists x[\forall y(P(x,y)]$, if we are given that $\exists x[\forall y(P(x,y)]$.

Grading: 0 pts if they say it's false. 0 pts if they say the two are equivalent so it's true. 3 pts automatically if they say it's true but never say they are equivalent. 2 pts for their explanation as to why this is the case. Judge partial as needed, though be lenient as this is probably pretty hard to write down on paper. Full credit should be given if someone comes up with an example that seems reasonable.

5) (12 pts) Prove or disprove: If n is an integer such that $n \equiv 1\ (mod 6)$, then $n^2 \equiv 1\ (mod\ 24)$. (Note: You may use the result that we previously proved in homework, namely, for any integer a, $a(3a+1)$ is an even integer.

The assertion is true. Let n be an arbitrary integer such that $n \equiv 1\ (mod 6)$.
It follows that 6 | (n - 1) and there exists an integer x such that n - 1 = 6x.
Consequently, n = 6x + 1. Note that since x is an integer x(3x+1) is an even integer.
Thus, there exists another integer y such that x(3x+1) = 2y:

$$n^2 = (6x + 1)^2 = 36x^2 + 12x + 1 = 12x(3x + 1) + 1 = 12(2y) + 1 \equiv 1\ (mod 24)$$

Grading: Many ways to do this, so adjust your grading accordingly!

3 pts - setting n to 6x + 1 for some integer x
3 pts - properly squaring 6x + 1
2 pts - factoring out 12 from the first two terms
2 pts - recognizing x(3x+1) as even in some way shape or form
2 pts - reducing the quantity or showing that it's equivalent to 1 mod 24.

6) (10 pts) Prove or disprove the following assertion about finite sets A and B, taken from the positive integers (Note: $\wp$ indicates Power Set.):

$$\wp(A) - \wp(B) \subseteq \wp(A - B).$$

This is false. Let A = {1, 2} and B = {1}. Then we have the following sets:

$$A - B = \{2\},$$
$$\wp(A - B) = \{\emptyset, \{2\}\},$$
$$\wp(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\},$$
$$\wp(B) = \{\emptyset, \{1\}\},$$
$$\wp(A) - \wp(B) = \{\{2\}, \{1,2\}\}$$

Thus, for this particular example, we see $\{1,2\} \in \wp(A) - \wp(B)$, but $\{1,2\} \notin \wp(A - B)$, proving that the statement is false for this specific pair of sets A and B and can not be true for all finite sets A and B.

Grading: 2 pts max for any proof, 3 pts for just saying it's false. 2 pts for explicitly stating values sets A and B. 3 pts if the values they pick are ACTUALLY a counter-example, 2 pts for explaining WHY those values are a counterexample.

7) (10 pts) Prove or disprove the following assertion about finite sets A, B, and C taken from the positive integers:

$$(A - C) - (B - C) \subseteq (A - B)$$

This assertion is true for all sets A, B and C. By definition of subset, we must prove that for an arbitrarily chosen value x such that $x \in (A - C) - (B - C)$, that $x \in (A - B)$. We use direct proof. Assume that $x \in (A - C) - (B - C)$.

By definition of set difference we have that $x \in (A - C) \wedge x \notin (B - C)$.

By definition of set difference applied to the first item, we have $x \in A \wedge x \notin C$.

By definition of set difference applied to the second item, we have $\overline{x \in B \wedge x \notin C}$.

Using DeMorgan's Law on this last deduction, we get $\overline{x \in B} \vee \overline{x \notin C}$. By definition of element of, this reduces to $x \notin B \vee x \in C$. Thus, at least one of these two things must be true.

But we already know that $x \notin C$, from a previous deduction. Thus, by Disjunctive Syllogism, it follows that $x \notin B$.

At this point we've deduced that $x \in A \wedge x \notin B$. By definition of set difference, it follows that $x \in A - B$, as desired, completing the proof.

Grading: Many, many ways to prove this. For this particular solution, map the points as follows:

2 pts for stating what needs to be proved.

2 pts for applying the set difference definition for the outer -.

2 pts for applying the set difference definition for both inner -.

3 pts for using the Disjunctive Syllogism reasoning (BUT THEY DON'T HAVE TO STATE THE REASON!!!) to obtain that x isn't in B.

1 pt for using the set difference definition to get to the conclusion.

For different solutions, please map the points similar to how these points are mapped.

8) (10 pts) Consider finite sets A, B and C where we know the cardinalities of the following sets:

$$|A \cap C| = 0$$
$$|A \cup B| = 20$$
$$|B \cap C| = 5$$
$$|A \cup B \cup C| = 22$$

Determine |C|.

Note that if $A \cap C$ is the empty set, so is $A \cap B \cap C$, because intersecting anything with the empty set yields the empty set.

Here is the Inclusion-Exclusion Principle for three sets. Use this to solve for |C|:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$|C| = |A \cup B \cup C| - |A| - |B| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$
$$|C| = |A \cup B \cup C| - (|A| + |B| - |A \cap B|) + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

Now, note that for two sets, we have $|A \cup B| = |A| + |B| - |A \cap B|$, which we can use for substitution. Thus, we can simplify the three terms in the parentheses as follows:

$$|C| = |A \cup B \cup C| - (|A \cup B|) + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

Next, recall that both $A \cap C$ and $A \cap B \cap C$ are empty for this particular problem, so we can substitute into each piece as follows:

$$|C| = 22 - 20 + 0 + 5 - 0 = 7$$

Grading: 2 pts for stating I/E for 3 sets, 3 pts for deduction that AintBintC is empty, 3 pts for deduction that we can plug in for |A|+|B|-|AintB| without knowing the individual values, 2 pts for the final answer and correct arithmetic.

9) (3 pts) Let $S = \{2x^2 + x - 6 = 0 | x \in Z\}$. (Note: Recall that Z is the set of integers.) Explicitly list each element that belongs to the set S. Put a circle around your final answer.

Factoring the given expression, we get (2x-3)(x+2) = 0, thus the set of all real numbers where the given equation is true is x = 3/2 or x = -2. But, 3/2 isn't an integer, so the set S only contains one integer, -2. Formally, S = {-2}.

Grading: 3 pts for the correct set, 2 pts for putting both 3/2 and -2 in it, 1 pt for either doing the quadratic or factoring but putting values in the set that don't actually satisfy the equation, 0 pts otherwise.

10) (1 pt) At what time in the afternoon does one develop a five o-clock shadow?
5 o'clock, specifically 5 pm. **(Grading - give to all)**