COT 3100 Fall 2017 Homework 9 Solutions

1) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4}.
In particular, let R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3) (4, 4)} and R_2 = {(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)}.

Determine the following:

a) Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R_1 and R_2.

Solution

a) R_1 is reflexive, because it contains all ordered pairs of the form (a, a) for all a in A.
R_1 is NOT irreflexive because it contains (1, 1).
R_1 is symmetric, for each ordered pair of the form (a, b), it also contains (b, a).
R_1 is not anti-symmetric since it contains both (1, 2) and (2, 1) and $1 \neq 2$.
R_1 is transitive, because for all ordered pairs (a, b) and (b, c) in R_1, (a, c) is also in R_1.
(To see this, note that if a = 1 or 2, b = 1 or 2 and if b = 1 or 2, c = 1 or 2. Similarly, if a = 3 or 4, then b = 3 or 4 and c = 3 or 4.)

b) R_2 is not reflexive because (1,1) is not in R_2.
R_2 is irreflexive because for all a in A, it doesn't contain (a, a).
R_2 is not symmetric because it contains (1, 2) but not (2, 1).
R_2 is anti-symmetric, for all (a, b) in R_2 with $a \neq b$, (b, a) is NOT in R_2.
R_2 is NOT transitive. It contains (1, 3) and (3, 4) but not (1, 4).

c) $R_1 \circ R_2$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)}.

d) $R_2 \circ R_1$ = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 4)}.

e) $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(2,4),(3,3),(3,4),(4,3)(4,4)\}$

f) $R_1 \cap R_2 = \{(1,2),(3,4)\}$

g) $r(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)\ (4,4)\}$
$s(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)\ (4,4)\}$
$t(R_1) = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3)\ (4,4)\}$
$r(R_2) = \{(1,1),(1,2),(2,2),(2,3),(3,3),(3,4),(4,4),(1,3),(2,4)\}$
$s(R_2) = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(1,3),(3,1),(2,4),(4,2)\}$
$t(R_2) = \{(1,2),(2,3),(3,4),(1,3),(2,4),(1,4)\}$

2) Let R be a relation on the set Z^+ defined as follows:

R = {(a, b) | $\exists c \in Fibonacci\ such\ a + b = c\ or\ a - b = c$}

Let the set Fibonacci be the set of positive integers that are Fibonacci numbers.

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

Solution

R is not reflexive because (2, 2) is not in R. Namely, neither 2 + 2 nor 2 - 2 are positive Fibonacci numbers.

R is not irreflexive because (1, 1) is in R because 1 + 1 = 2 and 2 is a Fibonacci number.

R is not symmetric because (6, 1) is in R (since 6 - 1 = 5 and 5 is a Fibonacci number, but (1, 6) is not in R because either 1 + 6 = 7 nor 1 - 6 = -5 are positive Fibonacci numbers.

R is not anti-symmetric because (2, 3) and (3, 2) are both in R because 2 + 3 = 5 (a Fibonacci number) and 3 - 2 = 1 (a Fibonacci number).

R is not transitive. We can use the same pairs in the previous example: both (2, 3) and (3, 2) are in R but (2, 2) is not in R since neither 2 + 2 = 4 nor 2 - 2 = 0 are positive Fibonacci numbers.

3) Let b(n) equal the number of bits set to 1 in the binary representation of the positive integer n. Prove that the relation, R, defined below over the ***set of positive integers in between 1 and 1024***, inclusive, is an equivalence relation. Into how many equivalence classes does R partition the set described? Explicitly list all of the members of the following equivalence classes: [2] and [520]. Let the set X be the largest of the equivalence classes. What is the smallest integer that belongs to X?

R = {(x, y) | $b(x) = b(y)$}

Solution

For any integer, x, the number of bits in its binary representation is equal to the number of bits in its binary representation, thus, (x, x) belongs in the relation R.

The relation is symmetric. If x has the same number of bits set to 1 in its binary representation, y ALSO has the same number of bits set to 1 as x. Thus, if (x, y) is in R, so is (y, x).

The relation is transitive. If x and y have the same number of bits set to 1 in their binary representations and y and z have the same number of bits set to 1 in their binary representation, then the number of bits set to 1 in x and z must also be the same. Thus, if (x, y) and (y, z) are in R, (x, z) must also be.

[2] = {1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024}

[520] = {3, 5, 6, 9, 10, 12, 17, 18, 20, 24, 33, 34, 36, 40, 48, 65, 66, 68, 72, 80, 96, 129, 130, 132, 136, 144, 160, 192, 257, 258, 260, 264, 272, 288, 320, 384, 513, 514, 516, 520, 528, 544, 576, 640, 768}

Basically, the first of these sets contains each binary number with 1 bit on from 1 to 1024 and the second contains each binary number with 2 bits on from 1 to 1024. Notice that there are $_{10}C_2$ ways to choose 2 bits out of 10 and that all of our numbers (except 1024) has 10 or fewer bits, so we can just cycle through the 45 ways of choosing any 2 bits from the 10 least significant bits to set on to generate the list.

The largest equivalence class is the set with 5 bits on, since $_{10}C_5$ is larger than $_{10}C_k$ for all other k from 0 to 10. The smallest member of this equivalence class is 31, since 31 has the five least significant bits set to 1.

4) Let R be a relation on the set Z^+ defined as follows:

$R = \{(a, b) \mid |a - b| \le 3\}$

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

Solution

R is reflexive. For all integers a, $|a - a| = 0 < 3$, so it follows that (a, a) is in R for all integers a.

R is not irreflexive since (1, 2) is in R.

R is symmetric. If (a, b) is in R, then $|a - b| \le 3$. Since $|b - a| = |(-1)(a-b)| = |-1||a-b| = |a - b| \le 3$, it follows that (b, a) is also in R.

R is not anti-symmetric since (2, 3) is in R, (3, 2) is in R and $2 \neq 3$.

R is not transitive since (2, 4) is in R and (4, 6) is in R, but (2, 6) isn't in R. This is similar to the fact that person B can borrow person A's clothes (but those clothes might be a bit tight) and that person C can borrow person B's clothes (but again, they'll be a bit tight), but that it would be too much of a stretch, literally, for person C to borrow person A's clothes.

5) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7} contain the ordered pairs (2, 3), (3, 3) and (6, 6)?

Solution
Anti-symmetric relationships can contain ordered pairs of the form (a, a), but don't "have" to. There are seven such ordered pairs, of which we are *forced* to include 2. We are free to do what we want with the other 5. There are 2^5 combinations of ways to include/not include ordered pairs of the form (a, a).

Anti-symmetric relationships are not allowed to contain two ordered pairs of the form (a, b) and (b, a) where a ≠ b. There are $(7^2 - 7)/2 = 21$ such pairs of ordered pairs. Normally, for these ordered pairs, there are three possible arrangements: none or one of the ordered pairs are included in the relation. For 20 of the 21 pairs, we can make any of these 3 choices, for 3^{20} number of combinations of these ordered pairs. For the 21^{st} pair of {(2, 3), (3, 2)}, since we are forced to include (2, 3), it forces us NOT to include (3, 2), thus we have no freedom of choice for this pair of ordered pairs.

It follows that the desired number of relations is $2^5 3^{20}$.

6) Let $f(x) = x^2 + 2x - 35$ with a domain of all real $x \in (-\infty, -1]$. Prove that f is injective. What is the range of f? (You may either use Calculus or complete the square to prove your answers.)

Solution
To prove the function is injective, we must show that if f(a) = f(b), then a = b. For an arbitrary a and b, both in the interval $(-\infty, -1]$, let f(a) = f(b):

$$f(a) = f(b)$$
$$a^2 + 2a - 35 = b^2 + 2b - 35$$
$$a^2 - b^2 + 2a - 2b = 0$$
$$(a - b)(a + b) + 2(a - b) = 0$$
$$(a - b)(a + b + 2) = 0$$

Thus, we must have either $a - b = 0$ or $a + b + 2 = 0$. Remember that both $a \le -1$ and $b \le -1$. The only values of a and b that satisfy these inequalities AND satisfy the second equation are a = b = -1. (If either a or b is strictly less than -1, then the second equation CAN NOT be satisfied.) Notice that in this case a = b. Alternatively, a - b = 0, which ALSO means that a = b. Thus, we've proven that if f(a) = f(b), then a = b must follow.

7) Find $f^{-1}(x)$ for the function given in question #6.

Solution
$f(x) = x^2 + 2x - 35$

To find the inverse function, "swap" x and y and solve for y:

$x = y^2 + 2y - 35$
$x = y^2 + 2y + 1 - 35 - 1$
$x = (y + 1)^2 - 36$
$x + 36 = (y + 1)^2$

When we take the square root of both sides, we must note that the domain of the original function was values of x less than or equal to -1. Thus now corresponds to the range of the inverse function. If we want y to be -1 or less, when we take the square root of both sides, we must choose the negative sign, thus for the known range of the inverse function, we must have:

$\sqrt{(y + 1)^2} = -\sqrt{x + 36}$
$y + 1 = -\sqrt{x + 36}$
$y = -1 - \sqrt{x + 36}$

It follows that the desired inverse function is $f^{-1}(x) = -1 - \sqrt{x + 36}$.

8) Let A be a set of 12 elements and B be a set of 20 elements. How many functions can be defined with the domain of A and the co-domain of B?

Solution
Each item in A must map to precisely 1 of 20 items in B. The mapping for a single item can be done in 20 ways, and each single item in A is independent of the rest. Thus, the total number of possible functions is 20^{12}, using the multiplication principle (repeatedly).

9) Let $f(x) = 5x^2 + 2x - 7$ and $g(x) = 3x + 4$. Determine f(g(x)) and g(f(x)).

Solution
$$\begin{aligned} f\big(g(x)\big) = f(3x + 4) &= 5(3x + 4)^2 + 2(3x + 4) - 7 \\ &= 5(9x^2 + 24x + 16) + 6x + 8 - 7 \\ &= 45x^2 + 120x + 80 + 6x + 8 - 7 \\ &= 45x^2 + 126x + 81 \end{aligned}$$

$$g\big(f(x)\big) = g(5x^2 + 2x - 7) = 3(5x^2 + 2x - 7) + 4 = 15x^2 + 6x - 17$$

10) Let f(x) = 2x + 3. Let $f^n(x)$ to be the function f composed with itself n times. (For example, $f^3(x) = f(f(f(x)))$.) Using trial and error, conjecture a guess for $f^n(x)$ and use mathematical induction to prove that guess.

Solution

Let's work out $f^2(x)$, $f^3(x)$ and $f^4(x)$ and see if we can find a pattern:

$f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 = 4x + 9$
$f^3(x) = f(4x + 9) = 2(4x + 9) + 3 = 8x + 21$
$f^4(x) = f(8x + 21) = 2(8x + 21) + 3 = 16x + 45$

It's clear that the coefficient to x is just 2^n. Also, it's fairly easy to see the mechanism by how this works (we double the coefficient of x each time with the 2x...)

The other pattern for the constant is more subtle. We can factor out a 3 from each of the constant terms (for f(x), $f^2(x)$, $f^3(x)$ and $f^4(x)$) to get the following sequence 1, 3, 7, 15. Hopefully now the pattern is more clear: The constant term is $3(2^n - 1)$.

Thus, our conjecture is as follows: For all positive integers n, $f^n(x) = 2^n x + 3(2^n - 1)$. We prove our conjecture via mathematical induction:

Base case: n = 1. LHS = $f^1(x) = 2x + 3$. RHS = $2^1 x + 3(2^1 - 1) = 2x + 3$.
Thus, the formula holds for the base case.

Ind. hypothesis: Assume for an arbitrary positive integer n = k that $f^k(x) = 2^k x + 3(2^k - 1)$.

Inductive step: Prove for n = k + 1 that $f^{k+1}(x) = 2^{k+1}x + 3(2^{k+1} - 1)$.

$f^{k+1}(x) = f(f^k(x))$
$= f(2^k x + 3(2^k - 1))$, using the inductive hypothesis
$= 2\left(2^k x + 3(2^k - 1)\right) + 3$, applying the function f.
$= 2^{k+1}x + 6(2^k - 1) + 3$
$= 2^{k+1}x + 6(2^k) - 6 + 3$
$= 2^{k+1}x + 3(2^{k+1}) - 3$, combining powers of 2
$= 2^{k+1}x + 3(2^{k+1} - 1)$

This completes the inductive step. We can now conclude that for all positive integers n,

$$f^n(x) = 2^n x + 3(2^n - 1).$$

11) Give a summary of the life and mathematical contributions of Leonard Euler. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.

Sample Write Up

Leonard Euler is widely regarded as the most prolific mathematician in history. Euler lived entirely during the 18^{th} century, passing away in 1783. Euler was a precocious child who had an amazing memory and an aptitude for mathematics. He devoted most of his life to be an academic studying mathematics and related fields and made enormous contributions. In fact, after he passed away, a mathematical journal spent the next 48 years publishing the work he didn't get to publish while he was alive!!!

At the age of 14 he studied at the University of Basel under Johann Bernoulli. Bernoulli rarely took a liking to students but was impressed with Euler's talent and met with him on a regular basis to talk about all things mathematics. After completing school, Euler took a faculty post in St. Petersburg, Russia where Bernoulli's son, Daniel was also teaching. When Daniel left for a job in Switzerland, Euler was able to take his position as the mathematics chair at St. Petersburg.

Over the course of his career, Euler had various posts at universities and contributed to many branches of mathematics. One of Euler's classic results (derived earlier in his career) was $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, the solution to the so-called Basel Problem. (Incidentally, Euler was born near Basel.) Euler is also well-known for discovering the following relationship between complex numbers and e: $e^{i\theta} = cos\theta + isin\theta$. (And, of course, the constant e is named after Euler!) A result from graph theory that is due to Euler is that the vertices (V), edges (E), and faces(F) of a planer graph satisfy the equation V - E + F = 2. Another interesting result in graph theory due to Euler is proving exactly when a graph has an Euler Circuit (precisely when the degree of each vertex is even).

Even though Euler was blind for the last ten years of life, he continued publishing via scribes (well-versed in mathematics) who would write down what he said. His mental arithmetic at this late stage was so good that he could point out errors the scribes made many digits past the decimal without writing anything down!

Source: Euler - The Master of Us All (Dunham, ISBN: 0-88385-300-0)