COT 3100 Fall 2017 Homework #2 Solutions

1) Let n be an odd integer. Prove that $n^2 - 1$ is divisible by 8. After you prove this formally, attempt to give an intuitive reason why this turns out to be true similar to the intuitive reason that integers of the form k(k+1), when k is an integer, are even.

Solution

We'll use direct proof. Let n be an arbitrary odd integer. Then there exists an integer k such that n = 2k + 1. Let's consider the expression $n^2 - 1$:

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4(k^2 + k) \\ &= 4k(k+1) \end{aligned}$$

Now, let's do an aside and consider the expression k(k+1) for an arbitrary integer k. Let's do a quick proof by cases here. K is either even or odd.

If k is even, there exists an integer x such that k = 2x, and then we have

k(k+1) = (2x)(2x+1) = 2(x(2x+1)), which shows that the expression is even, since x is an integer.

If k is odd, there exists an integer x such that k = 2x, and then we have

k(k+1) = (2x)(2x+1) = 2(x(2x+1)), which shows that the expression is even, since x is an integer.

Now, we can return to our proof knowing that if k is an integer, then k(k+1) is an even integer.

Thus, we know there exists some integer c such that k(k+1) = 2c. So, the last step of our proof looks like:

$$\begin{aligned} &= 4k(k+1) \\ &= 4(2c) \\ &= 8c, \text{ proving that the original expression is divisible by 8.} \end{aligned}$$

Intuitively, the expression is even because any even times any integer is even, and given two consecutive integers, exactly one of them must be even, since evens and odds toggle back and forth.

2) Prove the following statement using (a) direct proof and (b) proof by contradiction.

If n is an even integer, then $3n+2$ is even.

Solution

Direct proof: If n is even, there exists an integer c such that $n = 2c$. Now, let's look at the expression at hand:

$3n + 2 = 3(2c + 2) = 6c + 6 = 2(3c + 3)$, since c is an integer, $3c + 3$ is as well and our original expression is even.

Proof by contradiction: Assume to the contrary that $3n+2$ is odd. Then there exists an integer c such that $3n + 2 = 2c + 1$:

$3n + 2 = 2c + 1$
$3n = 2c - 1$

Since we know that n is even, there exists an integer d such that $n = 2d$. Plugging in we get:

$3(2d) = 2c - 1$
$6d = 2c - 1$
$6d - 2c = 1$
$2(3d - c) = 1$

Since c and d are integers, so is $3d - c$, thus the left hand side of the equation is even, but 1 is odd. It's impossible for an even integer to equal 1. Thus, we have a contradiction. WE must conclude that the original assumption, that $3n + 2$ is odd was incorrect, proving that $3n + 2$ is even, as desired.

3) Use direct proof to show that every odd integer can be expressed as the difference of two perfect squares. (Note: A perfect square is simply any integer of the form n^2, where n is an integer.)

Solution

Let n be an arbitrary odd integer. Then there exists an integer c such that $n = 2c + 1$. Consider the two perfect squares, $(c+1)^2$ and c^2. Their difference is:

$(c+1)^2 - c^2 = c^2 + 2c + 1 - c^2 = 2c + 1$.

Thus, for any odd integer, n, we can obtain the appropriate value for c, and using it, we can construct a pair of perfect squares whose difference is n. (For example, if we try $n = 17$, the corresponding value of c is $c = 8$. Then $9^2 - 8^2 = 81 - 64 = 17$, as desired.)

4) Given a set of real numbers a_1, a_2, …, a_n, let their average be b. Prove that there exists at least one number in the set that is greater than or equal to b. (Hint: use proof by contradiction!)

Solution

Assume to the contrary, that for all a_i, $1 \le i \le n$, that a_i, namely that all numbers in the set are strictly below the average. By definition of average, we have:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = b$$

$$a_1 + a_2 + \cdots + a_n = bn$$

Now, let's use are assumed information that each element in the set is strictly less than b:

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &< b + b + b + .. + b \text{ (n times)} \\ &= bn \end{aligned}$$

In short, using this information, we've proven that the sum of the values in our set is strictly less than bn, but this contradicts our information that the sum precisely equals bn. It follows that our incorrect step was our assumption that all values in the set were strictly less than the average. We can conclude that at least one value is either equal to or greater than the average.

5) Let S = {2, 3, 5} and T = {1, 2, 4, 6}. Explicitly list the members of the following sets: $S \cup T$, $S \cap T$. $S - T$, $S \times T$, $T \times S$, $\wp(S)$ and $\wp(T)$.

Solution

$S \cup T = \{1,2,3,4,5,6\}$

$S \cap T = \{2\}$

$S - T = \{3, 5\}$

$S \times T = \{(2,1), (2,2), (2,4), (2,6), (3,1), (3,2), (3,4), (3,6), (5,1), (5,2), (5,4), (5,6)\}$

$T \times S = \{(1,2), (1,3), (1,5), (2,2), (2,3), (2,5), (4,2), (4,3), (4,5), (6,2), (6,3), (6,5)\}$

$\wp(S) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2,3\}, \{2,5\}, \{3,5\}, \{2,3,5\}\}$

$\wp(T)$
$= \{\emptyset. \{1\}, \{2\}, \{4\}, \{6\}, \{1,2\}, \{1,4\}, \{1,6\}, \{2,4\}, \{2,6\}, \{4,6\}, \{1,2,4\}, \{1,2,6\}, \{1,4,6\}, \{2,4,6\}, \{1,2,4,6\}\}$

6) Let *A*, *B*, and *C* be sets. Show that

$$(A - B) - C = (A - C) - (B - C).$$

Hint: Start with the right side of the equation and use the Set Laws and Definitions to prove that this set is equivalent to the left side of the equation.

Solution

$(A - C) - (B - C) = (A \cap \bar{C}) - (B \cap \bar{C})$, using the definition of set difference
$= (A \cap \bar{C}) \cap \overline{B \cap \bar{C}}$, using the definition of set difference
$= (A \cap \bar{C}) \cap (\bar{B} \cup \bar{\bar{C}})$, using DeMorgan's Law
$= (A \cap \bar{C}) \cap (\bar{B} \cup C)$, using Double Negation
$= \big((A \cap \bar{C}) \cap \bar{B}\big) \cup ((A \cap \bar{C}) \cap C)$, using the Distributive Law
$= \big((A \cap \bar{B}) \cap \bar{C}\big) \cup (A \cap (\bar{C} \cap C))$, using the Comm./Assoc. Laws
$= \big((A - B) \cap \bar{C}\big) \cup (A \cap (\emptyset))$, using def Set Diff, Inverse Law
$= \big((A - B) - C\big) \cup (\emptyset)$, using def Set Diff, Domination Law
$= \big((A - B) - C\big)$, using the Identity Law

7) Give a summary of the life and mathematical contributions of Srinivasa Ramanujan. Please aim for a length of roughly 200 - 400 words. ***Your summary must be typed.*** Please state the sources you used in writing your summary and please don't just copy the blurb from the textbook… (If you loved writing this summary, you should watch the movie, The Man Who Knew Infinity.)

Sample Write Up (based on the Wikipedia entry for Ramanujan)
Srinivasa Ramanujan is to mathematics what many consider Mozart to be for music. With minimal formal training in mathematics, Ramanujan derived some of the most complicated known mathematics and even new theorems, completely on his own.

Born in a small village outside of Madras, India in 1887, Ramanujan very quickly showed his affinity for mathematics. By age 11, soon after he had been introduced to formal mathematics in school, he was able to exhaust all of the knowledge of two college boarders who lived at his house. At the age of 16, he obtained a copy of a pure mathematics book and devoured its contents. Using this book as a starting point, he calculated the Euler-Masheroni constant to 15 decimal places. Soon after, Ramanujan attempted to attend college, but due to the fact that all he cared about was mathematics, he earned poor grades in other subjects. After completing school, he spent most of free time doing mathematics, but ultimately had to get a job. He landed a job as a clerk in a revenue department. Upon seeing Ramanujan's notebooks, his boss realized that he didn't deserve such a boring job.

Ramanujan's boss showed the notebooks to other mathematicians in the area. Eventually, these notebooks circulated among a few British mathematicians, several of whom felt that though Ramanujan may have some talent, he lacked formal training and rigor in his proofs. Finally, however, two famous British mathematicians, G. H. Hardy and J. E. Littlewood, felt that Ramanujan's genius deserved to be discovered by others. They offered to have Ramanujan come to England to do mathematical research with them. In 1914, at the age of 27, Ramanujan took a ship from India to England to join Hardy and Littlewood. In working with Hardy and Littlewood, they improved the rigor of the proofs of Ramanujan's results, as often times, Ramanujan relied on his instincts. . In 1918, he was elected a Fellow of the Royal Society, one of the youngest mathematicians to do so and only the second Indian ever, at the time.

Thoughout his life, Ramanujan experienced health problems. Though it was typically thought that he died of tuberculosis, some now believe that he actually died from amoebiasis, a treatable disease that was widespread in Madras at the turn of the 20th century. Ramanujan passed away on April 26, 1920. In six short years of work with Hardy and Littlewood, he produced a great quantity of results. Some of the well-known results he proved dealt with the problem of partitions, the number of ways to divide a postive integer into a set of positive integers. (For example, 4 can be partitioned as follows: 4, 2+2, 1+3, 1+1+2, and 1+1+1+1.), the Bernoulli numbers and coming up with series that can be used to generate many digits of π quickly.