COT 3100 Exam #2: Number Theory, Induction, Solutions
10/24/2017

1) (8 pts) Convert 6987 in base ten to hexadecimal (base 16).

Solution

16 | 6987
16 | 436 R 11 (B)
16 | 27 R 4
16 | 1 R 11 (B)
0 R 1

$1B4B_{16}$

Grading: 1 pt for each quotient (except 0), 1 pt for each remainder, 1 pt for reading backwards. Give 2/8 max if they don't use the method shown in class. (Most who do something else will guess powers of 16 to subtract out...)

2) (10 pts) Let a and b be integers such that 7 | (2a + 3b). Prove that 7 | (13a + 16b).

Solution

Since 7 | (2a + 3b), there exists an integer c such that 2a + 3b = 7c.

$$\begin{aligned} 13a + 16b &= 7a + 6a + 7b + 9b \\ &= 7(a + b) + 3(2a + 3b) \\ &= 7(a + b) + 3(7c) \\ &= 7(a + b + 3c) \end{aligned}$$

Since a, b and c are integers, so is a+b+3c. It follows that 13a + 16b is divisible by 7, as desired.

Grading: 2 pts for any attempt to rewrite 13a + 16b, 2 pts if that attempt has either a multiple of 7a or 7b, 2 more pts if a 7 is factored out anywhere, 2 pts for any substitution for 2a+3b as a multiple of 7, 2 pts for completing the proof.

3) (15 pts) Determine 59^{-1} mod 203. Please express your answer as an integer in between 0 and 202. In order to earn full credit you must use the Extended Euclidean Algorithm.

Solution

First, run the Euclidean Algorithm:

203 = 3 x 59 + 26
59 = 2 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

Start with the last equation and run the Extended Euclidean Algorithm:

5 - 2 x 2 = 1
5 - 2(7 - 5) = 1
5 - 2 x 7 + 2 x 5 = 1
3 x 5 - 2 x 7 = 1
3(26 - 3 x 7) - 2 x 7 = 1
3 x 26 - 9 x 7 - 2 x 7 = 1
3 x 26 - 11 x 7 = 1
3 x 26 - 11(59 - 2 x 26) = 1
3 x 26 - 11 x 59 + 22 x 26 = 1
25 x 26 - 11 x 59 = 1
25(203 - 3 x 59) - 11 x 59 = 1
25 x 203 - 75 x 59 - 11 x 59 = 1
25 x 203 - 86 x 59 = 1

Take this equation mod 203 and we have:

-86 x 59 ≡ 1 mod 203

It follows that the desired modular inverse is -86 ≡ **117 (mod 203).**

Grading: 5 pts Euclidean, 8 pts Extended, 1 pt to get -86, 1 pt to map to 117.

4) (12 pts) Let a = $2^3 3^5 5^2 7^1$ and b = $3^3 5^6 7^2$. How many divisors does a have? How many divisors does b have? Express the greatest common divisor of a and b in prime factorized form. Express the least common multiple of a and b in prime factorized form.

Solution

Number of Divisors of a = 4 x 6 x 3 x 2 = 24 x 6 = **144.** (Just take product of 1 plus each exponent.)

Number of Divisors of b = 4 x 7 x 3 = **84.**

GCD(a, b) = $\mathbf{3^3 5^2 7^1}$**.** (Just take the minimum of each corresponding exponent.)

LCM(a, b) = $\mathbf{2^3 3^5 5^6 7^2}$**.** (Just take the maximum of each corresponding exponent.)

Grading: 3 points for each part, largely no partial credit, only give partial credit (2/3) if there is one arithmetic error when multiplying out the correct terms or taking a max/min of an exponent.

5) (15 pts) Using induction on n, prove for all non-negative integers n that 9 | (2^{2n} + 6n - 1).

Solution

Base case: n= 0, Given quantity is $2^{2(0)} + 6(0) - 1 = 2^0 + 0 - 1 = 1 - 1 = 0$. Since 0 is divisible by 9 the base case holds.

Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that 9 | (2^{2k} + 6k - 1). Namely, there exists an integer c such that 2^{2k} + 6k - 1) = 9c.

Inductive step: Prove for n = k+1 that 9 | ($2^{2(k+1)}$ + 6(k+1) - 1).

$$\begin{aligned}
2^{2(k+1)} + 6(k+1) - 1 &= 2^{2k+2} + 6k + 6 - 1 \\
&= 2^2 2^{2k} + 6k + 6 - 1 \\
&= 4(2^{2k}) + 4(6k) - 3(6k) + 9 - 4 \\
&= 4(2^{2k} + 6k - 1) - 18k + 9 \\
&= 4(9c) - 9(2k - 1), \text{ using Inductive Hypothesis} \\
&= 9(4c - 2k + 1)
\end{aligned}$$

Since c and k are integers, it follows that 4c - 2k + 1 is also an integer. Thus, we can conclude that $2^{2(k+1)}$ + 6(k+1) - 1 is divisible by 9, completing the inductive step. It follows that the original statement is true for all non-negative integers n.

Grading: Base case - 2 pts, IH - 2 pts, IS - 2 pts, 1 pt factor out 2^2, 2 pts generate 4(6k) or equivalent, 2 pts generate -4, 2 pts use IH, 2 pts complete. Map points as necessary for different ways to express the quantity.

6) (10 pts) Using induction on n, prove for all positive integers n that $\sum_{i=1}^{n} i^2 \leq n^3$.

Solution

Base case: n=1 LHS = $\sum_{i=1}^{1} i^2 = 1$, RHS = 1^3 = 1,
so the base case holds as the LHS≤RHS for n=1.

Inductive Hypothesis: Assume for an arbitrary integer n = k that $\sum_{i=1}^{k} i^2 \leq k^3$.

Inductive Step: Prove for n = k+1 that $\sum_{i=1}^{k+1} i^2 \leq (k+1)^3$.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$\leq k^3 + k^2 + 2k + 1$, using Inductive Hypothesis.

$\leq k^3 + 3k^2 + 3k + 1$, since k > 0, 1 < 3, and 2 < 3.

$= (k+1)^3$

Grading: 1 pt BC, 2 pts IH, 2 pts IS, 1 pt split sum, 2 pts use IH, 1 pt replace, 1 pt conclude

7) (16 pts) Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{1} iH_i = 1(H_1) = 1$, RHS = $\frac{1(1+1)}{2} H_1 - \frac{1(1-1)}{4} = H_1 = 1$

Thus the equation is true for n = 1.

Inductive Hypothesis: Assume for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k} iH_i = \frac{k(k+1)}{2} H_k - \frac{k(k-1)}{4}$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} iH_i = \frac{(k+1)(k+2)}{2} H_{k+1} - \frac{k(k+1)}{4}$$

$$\sum_{i=1}^{k+1} iH_i = (\sum_{i=1}^{k} iH_i) + (k+1)H_{k+1}$$

$= \frac{k(k+1)}{2} H_k - \frac{k(k-1)}{4} + (k+1)H_{k+1}$, using IH

$$= \frac{k(k+1)}{2} (H_{k+1} - \frac{1}{k+1}) - \frac{k(k-1)}{4} + \frac{2(k+1)H_{k+1}}{2}$$

$$= \frac{k(k+1)H_{k+1}}{2} - \frac{k}{2} - \frac{k(k-1)}{4} + \frac{2(k+1)H_{k+1}}{2}$$

$$= \frac{(k^2+k+2k+2)H_{k+1}}{2} - \frac{2k}{4} - \frac{k^2-k}{4}$$

$$= \frac{(k^2+3k+2)H_{k+1}}{2} - (\frac{k^2+k}{4})$$

$$= \frac{(k+1)(k+2)H_{k+1}}{2} - \frac{k(k+1)}{4}$$

Grading: 2 pts BC, 2 pts IH, 2 pts IS, 1 pt split sum, 2 pts plug in IH, 2 pts remove H_k, 2 pts facoring out H_{k+1}, 3 pts ending algebra (mult out and factoriing)

8) (12 pts) Define the sequence t_n as follows: $t_0 = 7$, $t_1 = 10$, $t_n = 3t_{n-1} - 2t_{n-2}$, for all integers $n \geq 2$. Prove, using strong induction on n with 2 base cases, that for all non-negative integers n,

$$t_n = 4 + 3(2^n).$$

Solution

Base cases: $n = 0$, LHS $= t_0 = 4 + 3(2^0) = 4 + 3 = 7$, RHS $= 7$, holds for $n = 0$

$n = 1$, RHS $= t_1 = 4 + 3(2^1) = 4 + 6 = 10$, RHS $= 10$, holds for $n = 1$

Inductive Hypothesis: Assume for all non-negative integers n less than or equal to an arbitrarily chosen positive integer k that $t_n = 4 + 3(2^n)$.

Inductive Step: Prove for $n = k+1$ that $t_{k+1} = 4 + 3(2^{k+1})$.

$t_{k+1} = 3t_k - 2t_{k-1}$, using given recurrence

$= 3(4 + 3(2^k)) - 2(4 + 3(2^{k-1}))$

$= 12 + 9(2^k) - 8 - 6(2^{k-1})$

$= 9(2^k) -3(2)(2^{k-1}) + 4$

$= 9(2^k) - 3(2^k) + 4$

$= 6(2^k) + 4$

$= 3(2)(2^k) + 4$

$= 3(2^{k+1}) + 4$

This proves the inductive step, this the given formula holds for all non-negative integers n.

Grading: 2 pts base cases, 2 pts IH, 1 pt IS, 2 pts plug in recurrence, 2 pts plugging into both IHs, 3 pts finishing algebra

9) (2 pts) How many losses has the undefeated UCF Knights football team suffered in the 2017 season?

Zero **(Grading: Give to all)**