Fall 2017 COT 3100 Exam #1 (9/26/2017) Solutions

1) (8 pts) Complete filling out the truth table below that evaluates the logical expression:

$$(p \wedge \bar{q}) \vee (r \rightarrow \bar{q}).$$

p	q	r	$p \wedge \bar{q}$	$r \rightarrow \bar{q}$	$(p \wedge \bar{q}) \vee (r \rightarrow \bar{q})$
F	F	F	**F**	**T**	**T**
F	F	T	**F**	**T**	**T**
F	T	F	**F**	**T**	**T**
F	T	T	**F**	**F**	**F**
T	F	F	**T**	**T**	**T**
T	F	T	**T**	**T**	**T**
T	T	F	**F**	**T**	**T**
T	T	T	**F**	**F**	**F**

Grading: 1 pt per row, only give credit for the row if it's completely correct.

2) (12 pts) Using the laws of logic, show that two following expressions are logically equivalent.

$$((\bar{p} \vee r) \wedge (\bar{r} \vee \bar{p})) \rightarrow (\bar{q} \wedge (r \vee \bar{q})) \qquad\qquad p \vee \bar{q}$$

Note: You may not use all of the rows shown below.

Step	Reason
1. $((\bar{p} \vee r) \wedge (\bar{r} \vee \bar{p})) \rightarrow (\bar{q} \wedge (r \vee \bar{q}))$	Given
2. $((\bar{p} \vee r) \wedge (\bar{p} \vee \bar{r})) \rightarrow (\bar{q} \wedge (\bar{q} \vee r)$	**Commutative Law (twice)**
3. $(\bar{p} \vee (r \wedge \bar{r})) \rightarrow (\bar{q} \wedge (\bar{q} \vee r))$	**Distributive Law**
4. $(\bar{p} \vee F) \rightarrow (\bar{q} \wedge (\bar{q} \vee r))$	**Inverse Law**
5. $\bar{p} \rightarrow (\bar{q} \wedge (\bar{q} \vee r))$	**Identity Law**
6. $\bar{p} \rightarrow \bar{q}$	**Absorption Law**
7. $\bar{\bar{p}} \vee \bar{q}$	**Implication Identity**
8. $p \vee \bar{q}$	**Double Negation**

Grading: 12 pts for any correct response, 8 pts if steps are correct but reasons are completely omitted, 1 pt off per incorrect step or reason, but you can't go below 0 and you can't take off for carry over mistake.

3) (12 pts) Use the rules of inference to make the following argument:

p
$p \rightarrow q$
$q \rightarrow (r \wedge t)$
$\bar{r} \vee s$
----
s

Step	Reason
1. p	**Premise**
2. $p \rightarrow q$	**Premise**
3. q	**Modus Ponens (1, 2)**
4. $q \rightarrow (r \wedge t)$	**Premise**
5. $r \wedge t$	**Modus Ponens (3, 4)**
6. r	**Conjunctive Simplification (5)**
7. $\bar{r} \vee s$	**Premise**
8. s	**Disjunctive Syllogism (6, 7)**

Grading: 12 pts for any correct response, 8 pts if steps are correct but reasons are completely omitted, 1 pt off per incorrect step or reason, but you can't go below 0 and you can't take off for carry over mistake.

4) (10 pts) With proof, prove or disprove the following statement:

$$\forall y \in Z^+ \exists x \in Q^+ [\frac{x}{y} = x^2 - x]$$

Note: The set Z^+ is the set of positive integers, and the set Q^+ is the set of positive rational numbers.

This is true. We must show for all positive integers y, there exists a positive rational value x, such that $\frac{x}{y} = x^2 - x$. We do this by solving the given equation for x:

$\frac{x}{y} = x^2 - x$ **Grading: 4 pts to solve for y instead of x, 1 pt for picking any x**

$0 = x^2 - x - \frac{x}{y}$ **2 pts max if say it's false**

$0 = x^2 - x(1 + \frac{1}{y})$ **up to 8 pts for solving for x, decide partial**

$0 = x(x - \left(1 + \frac{1}{y}\right))$ **2 pts for showing that 1 + 1/y is rational.**

Thus, there are two values for x for which the equation holds: x = 0 and $x = 1 + \frac{1}{y} = \frac{y+1}{y}$. Since we are just trying to prove something for all positive integers, y, it suffices to show that $\frac{y+1}{y}$ is a positive rational number when y is a positive integer. If y is a positive integer, then both y and y+1 are also. By definition of a positive rational number, (a positive integer divided by another positive integer), it follows that $\frac{y+1}{y}$ is a positive rational number. Thus, we've shown that for any positive integer y, there is a positive rational value x such that $\frac{x}{y} = x^2 - x$. Namely, that rational value is $x = \frac{y+1}{y}$.

5) (8 pts) Prove or disprove: If n is an odd integer, then $4 \mid (n^3 - 1)$.

The claim is false. Consider the case of n = 3. In this case, $n^3 - 1 = 3^3 - 1 = 26$, but 26 is NOT divisible by 4. (Note: The claim is true for all integers equivalent to 1 mod 4.)

Grading: 2/8 for any proof with a valid expansion of $(2c+1)^3$. 3/8 if they answer false give the expansion and claim that $8c^3 + 12c^2 + 6c$ isn't divisible by 4 without a specific counter-example. For an answer with a counter-example, give full credit if they specify n, calculate $n^3 - 1$ and show that it isn't divisible by 4. Take off credit of a counter-example if the communication isn't clear. 6 of 8 if they expanded, said it was false gave k but didn't give n, 7/8 if they gave n but didn't explicitly show the counter-example.

6) (10 pts) Use the Set Laws to show that the two following sets are equal:

$$A \cup (B \cap \bar{C}) \qquad\qquad \overline{\bar{A} \cap \bar{B}} \cap (A \cup (A \cap C) \cup \bar{C})$$

Note: You may not use all of the rows shown below.

Step	Reason
1. $\overline{\bar{A} \cap \bar{B}} \cap (A \cup (A \cap C) \cup \bar{C})$	**Given**
2. $(\bar{\bar{A}} \cup \bar{\bar{B}}) \cap (A \cup (A \cap C) \cup \bar{C})$	**De Morgan's Law**
3. $(\bar{\bar{A}} \cup \bar{\bar{B}}) \cap (A \cup \bar{C})$	**Absoprtion Law**
4. $(A \cup B) \cap (A \cup \bar{C})$	**Double Negation**
5. $A \cup (B \cap \bar{C})$	**Distributive Law**

Grading: 1 pt per step, 1 pt per reason corresponding to the ones up there. 10 pts for any correct response.

7) (12 pts) Prove or disprove the following assertion about sets A, B and C:

if $B \subseteq A \cap C$, then $A - C \subseteq A - B$.

The assertion is true. Namely, we must show that if x is an arbitrarily chosen element of A - C, then x is also an element of A - B. We use direct proof to show it.

Let x be an arbitrary element of A - C. (Grading: 2 pts)

By definition of set difference, $x \in A \wedge x \notin C$. (Grading: 2 pts)

By definition of set intersection, if $x \notin C$, then $x \notin A \cap C$, since automatically, if an item isn't in one set, it can not be in both that set and some other set. (Grading: 3 pts)

By definition of subset, if B is a subset of $A \cap C$, and x doesn't belong to $A \cap C$, we can conclude that x does not belong to B. (If x did belong to B, then B couldn't be a subset of $A \cap C$.) (Grading: 4 pts)

Since $x \in A \wedge x \notin B$, by definition of set difference, it follows that $x \in A - B$. (1 pt)

Grading: 1/12 for any attempt at a counter-example, 1/12 for any proof that starts off with x being an arbitrary element of B, 2/12 for any example that works as a proof, for proof by contradiction:

3 pts for setting up x in A-C that isn't in A-B. 4 pts for implications for sets B and C. 5 pts for finding contradiction and completing the proof

8) (10 pts) Prove of disprove the following assertion about sets A, B, C and D:

$$(A \cup B) \times (C \cap D) = (A \times C \cap B \times C) \cup (A \times D \cap B \times D)$$

This is false. Consider the following counter-example: A = {1}, B = {1}, C = {1}, D = {2}. In this counter-example, $\boldsymbol{A \cup B = \{1\}}$, $\boldsymbol{C \cap D = \emptyset}$, so $\boldsymbol{(A \cup B) \times (C \cap D) = \emptyset}$.

But, we have the following four Cartesian Products: $\boldsymbol{A \times C = \{(1,1)\}}$, $\boldsymbol{B \times C = \{(1,1))\}}$,
$\boldsymbol{A \times D = \{(1,2)\}}$, $\boldsymbol{B \times D = \{(1,2)\}}$.

It follows that $\boldsymbol{(A \times C \cap B \times C) \cup (A \times D \cap B \times D) = \{(1,1),(1,2)\}}$.

Thus, for this counter-example, we see that the left-hand side and right-hand side are unequal, so the claim can not be true for all sets A, B, C and D.

Grading: max 2 pts for any proof

1 pt for saying it's false.
2 pts for creating some sort of counter-example.
2 pts for specifying A, B, C and D separately.
2 pts for specifying A union B, C intersection D
3 pts for writing out Cartesian products that show the two sides aren't equal.

9) (12 pts) Consider finite sets A, B and C where we know the cardinalities of the following sets:

$$|A \cap C| = 7$$
$$|A \cap B \cap C| = 3$$
$$|A \cup B| = 20$$
$$|A \cap B| = 8$$
$$|B \cup C| = 22$$
$$|A \cup B \cup C| = 23$$

Determine |B|. (Note: Solutions that only use a Venn-Diagram to determine the answer will receive a maximum of 6/12 points. Only solutions that use a formal proof involving the Inclusion-Exclusion principle will earn full credit.)

First, let's set up the Inclusion-Exclusion Principle for sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$20 = |A| + |B| - 8$$
$$|A| + |B| = 28$$

Now, let's set up the Inclusion-Exclusion Principle for sets A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$23 = 28 + |C| - 8 - 7 - |B \cap C| + 3$$
$$|C| - |B \cap C| = 7$$

Now, set up the Inclusion-Exclusion Principle for sets B and C:

$$|B \cup C| = |B| + |C| - |B \cap C|$$
$22 = |B| + 7$, (note the substitution for $|C| - |B \cap C|$ from the I/E for 3 sets.)
$$|B| = 15$$

Note: Based on the given information $|B \cap C|$ is not uniquely determined, but $|B|$ is.

Grading: 6/12 for correct intuitive answer (without I/E), grade out of 6 if it's an incorrect intuitive result.

4 pts solving for $|A| + |B|$
4 pts solving for $|C| - |B \cap C|$
4 pts for finishing it up
Give partial as needed.

Note: There are probably a ton of ways to solve this, grade accordingly.

10) (1 pt) At what time in the afternoon does one develop a five o-clock shadow? **5 o'clock**
(1 pt give to everyone, even if they didn't answer it)