Fall 2017 COT 3100 Final Exam Part B Solutions

1) (15 pts) Using induction on n, prove for all integers n ≥ m, where m is a fixed positive integer, and j is another fixed non-negative integer such that j < m, that

$$\sum_{i=m}^{n} \binom{i}{j} = \binom{n+1}{j+1} - \binom{m}{j+1}$$

<u>Solution</u>

Base case: n = m, LHS = $\sum_{i=m}^{m} \binom{i}{j} = \binom{m}{j}$, RHS = $\binom{m+1}{j+1} - \binom{m}{j+1} = \binom{m}{j}$, via Pascal's Triangle identity. Thus, it follows that the base case holds for n = m.

Inductive hypothesis (IH): Assume for an arbitrary positive integer n = k, where k ≥ m, where n is a fixed positive integer and j is another fixed non-negative integer such that j < m that

$$\sum_{i=m}^{k} \binom{i}{j} = \binom{k+1}{j+1} - \binom{m}{j+1}$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=m}^{k+1} \binom{i}{j} = \binom{k+2}{j+1} - \binom{m}{j+1}$$

$$\begin{aligned}
\sum_{i=m}^{k+1} \binom{i}{j} &= \left(\sum_{i=m}^{k} \binom{i}{j}\right) + \binom{k+1}{j} \\
&= \binom{k+1}{j+1} - \binom{m}{j+1} + \binom{k+1}{j}, \text{ using the IH} \\
&= \binom{k+1}{j+1} + \binom{k+1}{j} - \binom{m}{j+1} \\
&= \binom{k+2}{j+1} - \binom{m}{j+1}, \text{ via the Pascal Triangle Identity}
\end{aligned}$$

This completes the inductive step. It follows that the identity is true for all positive integers n ≥ m.

Grading: base case - 2 pts, inductive hypothesis - 3 pts, inductive step - 3 pts, sum split - 2 pts, IH use - 2 pts, Pascal Triangle Identity - 3 pts

2) (10 pts) Consider the following algorithm to determine if an integer is divisible by 11 or not:

Go through the digits from least significant to most significant. Keep a counter that you start at zero and alternately add and subtract each digit from the counter. At the end, determine if the counter is divisible by 11. If it is, the original number is divisible by 11. If it isn't, the original number isn't divisible by 11.

Here is the algorithm being implemented on the number 76180258:

1. Add 8, counter is 8
2. Subtract 5, counter is 3
3. Add 2, counter is 5
4. Subtract 0, counter is 5
5. Add 8, counter is 13
6. Subtract 1, counter is 12.
7. Add 6, counter is at 18.
8. Subtract 7, counter is at 11.

Since 11 is divisible by 11, 76180258 is divisible by 11.

Prove that this algorithm works. Namely, show that the calculation prescribed is equivalent to the value of the original number mod 11. (A natural consequence of proving this is the correctness of the algorithm.)

<u>Solution</u>

Consider a k-digit number in base ten in its digit representation: n = $\sum_{i=0}^{k-1} d_i 10^i$. Now, consider this summation mod 11:

$$n = \sum_{i=0}^{k-1} d_i 10^i \equiv \sum_{i=0}^{k-1} d_i (-1)^i \ (mod\ 11)$$

In order for n to be divisible by 11, the n must be equivalent to 0 mod 11. This summation is nothing but

$$d_0 - d_1 + d_2 - d_3 + \cdots$$

This is precisely the alternately adding and subtracting digits starting with the least significant.

Thus, what the algorithm does is calculate the alternating sum given and then check for its remainder when divided by 11, if it's 0, the remainder of the original number when divided by 11 is 0 as well. If it's not zero, the remainder of the original number when divided by 11 is also not 0.

Grading: idea of viewing the sum of digits by contribution mod 11 - 4 pts, the appropriate reduction (10 = -1 mod 11) - 4 pts, conclusion - 2 pts