Fall 2017 COT 3100 Final Exam Part A Solution

1) (8 pts) It takes Bob six days to paint their townhouse, and it takes Carol ten days to paint their townhouse. For the purposes of this problem, assume a day is equal to 8 hours. (No one wants to paint "all" day!!!) They invite Seema over to help them paint their townhouse and complete the task in one day (eight hours). If Seema were painting the townhouse on her own, how many hours total would it have taken her to complete the task? Express your answer as a fraction in lowest terms.

Solution

Bob paints 1/6 of the house in a day and Carol paints 1/10 of the house in a day. Let Seema paint x of the house in a day. This gives us the equation:

$$\frac{1}{6}+\frac{1}{10}+x=1$$
$$\frac{16}{60}+x=1$$
$$x=\frac{44}{60}=\frac{11}{15}$$

Thus, Seema paints 11/15 of the house in a single day. It follows that she could paint the whole house in $\frac{15}{11}\,days=\frac{15}{11}\,days\times\frac{8\,hours}{1\,day}=\boldsymbol{\frac{120}{11}\,hours}$

Grading: 2 points for initial equation set up, 2 pts solving for variable, 2 pts calc days to paint, 1 pt convert to hours to paint, 1 pt lowest terms

2) (8 pts) Use an efficient technique via hand to calculate the remainder when 4^{20} is divided by 47. (You may follow the book's technique or come up with an ad hoc technique that follows valid mod rules to minimize work by hand.) Please show each step.

Solution

We will repeatedly square as shown in the text:

$4^2 \equiv 16 \bmod 47$
$4^4 \equiv (4^2)^2 \equiv 16^2 \equiv 256 \equiv 21 \bmod 47$
$4^8 \equiv (4^4)^2 \equiv 21^2 \equiv 441 \equiv 18 \bmod 47$
$4^{16} \equiv (4^8)^2 \equiv 18^2 \equiv 324 \equiv 42 \equiv -5 \bmod 47$
$4^{20} \equiv 4^{16}4^4 \equiv (-5)(21) \equiv -105 \equiv$ **36 mod 47**

Grading: Many ways to solve this. 5 pts for all intermediate results (grader decides partial), 3 pts for combining into the correct answer (2 pts for answer, 1 pt reduce to [0, 46].)

3) (15 pts) Find all integer solutions to the equation 51x + 192y = 39.

Solution

First, run the Euclidean Algorithm with 192 and 51:

192 = 3 x 51 + 39
51 = 1 x 39 + 12
39 = 3 x 12 + 3
12 = 4 x 3
gcd(192, 51) = 3

From here, we can proceed in a couple different ways - (a) divide the original equation by 3 and find all equations to that solution or (b) Run an adapted version of the Extended Euclidean with the equations above. Since (a) was shown previously in class, (b) will be presented here:

39 - 3 x 12 = 3
39 - 3(51 - 39) = 3
39 - 3 x 51 + 3 x 39 = 3
4 x 39 - 3 x 51 = 3
4(192 - 3 x 51) - 3 x 51 = 3
4 x 192 - 12 x 51 - 3 x 51 = 3
4 x 192 - 15 x 51 = 3

Now, multiply both sides by 13, since 39/3 = 13:

(4 x 13) x 192 - (15 x 13) x 51 = 3 x 13
52 x 192 - 195 x 51 = 39

It follows that an ordered pair that satisfies the equation is (-195, 52). We can rewrite this equation as follows by dividing through by the gcd(192, 51):

52 x 64 - 195 x 17 = 13

Notice that we can add any multiple of 17 to 52 and subtract the corresponding multiple of 64 from -195 to yield another solution. Namely, for any integer c, we have

(52 + 17c) x 64 + (-195 - 64c) x 17 = 13

Since 17 and 64 share no common factors, there are no smaller offsets that we can add and subtract respectively to generate other solutions. Thus, the entire solution set is as follows:

$$\{\,(-\mathbf{195} - \mathbf{64}\boldsymbol{c}, \mathbf{52} + \mathbf{17}\boldsymbol{c}) | \boldsymbol{c} \in \boldsymbol{Z}\,\}$$

Grading: 3 pts Euclidean, 6 pts Extended Euclidean, 2 pts extract one solution, 4 pts for finishing the problem (-1 if both + or both -, -2 if their offsets are 51 and 192)

4) (12 pts) Prove, for all positive integers n, using induction on n that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^n = \frac{1}{4}\begin{bmatrix} 5^n + 3 & 1 - 5^n \\ 3(1 - 5^n) & 3(5^n) + 1 \end{bmatrix}.$$

Solution

Base case: n = 1 - LHS = $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^1 = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$,

RHS = $\frac{1}{4}\begin{bmatrix} 5^1 + 3 & 1 - 5^1 \\ 3(1 - 5^1) & 3(5^1) + 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -12 & 16 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

Thus, the base case holds for n = 1.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^k = \frac{1}{4}\begin{bmatrix} 5^k + 3 & 1 - 5^k \\ 3(1 - 5^k) & 3(5^k) + 1 \end{bmatrix}.$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^{k+1} = \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 1 - 5^{k+1} \\ 3(1 - 5^{k+1}) & 3(5^{k+1}) + 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^{k+1} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^k \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^k + 3 & 1 - 5^k \\ 3(1 - 5^k) & 3(5^k) + 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2(5^k + 3) - 3(1 - 5^k) & -(5^k + 3) + 4(1 - 5^k) \\ 6(1 - 5^k) - 3((3)5^k + 1) & -3(1 - 5^k) + 4((3)5^k + 1) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2(5^k) + 6 - 3 + 3(5^k) & -5^k - 3 + 4 - 4(5^k) \\ 6 - 6(5^k) - (9)5^k - 3 & -3 + 3(5^k) + (12)5^k + 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5(5^k) + 3 & -(5)5^k + 1 \\ 3 - 15(5^k) & 15(5^k) + 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 1 - 5^{k+1} \\ 3(1 - 5(5^k)) & 3(5)(5^k) + 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 5^{k+1} + 3 & 1 - 5^{k+1} \\ 3(1 - 5^{k+1}) & 3(5^{k+1}) + 1 \end{bmatrix}$$

This completes the inductive step. It follows that the given statement is true for all positive integers n.

Grading: base case - 1 pt, inductive hyp - 2 pts, inductive step - 1 pt, split matrix - 1 pt, use IH - 2 pts, init multiply - 2 pts, simplify - 3 pts

5) (18 pts) Janice is trying to buy ornaments for her Christmas tree. The store she went to has 7 types of ornaments available. She'd like to buy a minimum of 5 ornaments and a maximum of 30 ornaments. In addition, she wants to buy at least two cupcake ornaments. There are only 10 ball ornaments available and 15 candy cane ornaments available. The store has more than 30 of all of the other five types of ornaments. In how many different ways can she buy ornaments satisifying all of her restrictions as well as the availability of ornaments at the store she went to? (Since this question is worth a lot of points, please carefully show and explain your work so that partial credit can be awarded.)

<u>Solution</u>

First, have Janice buy the two cupcake ornaments. Now, she has a minimum of 3 and maximum of 28 to purchase. From this point, let the number of each type of ornament she buys be a, b, c, d, e, f and g. Let a represent the number of ball ornaments and b represent the number of candy ornaments. Introduce a new variable h, to represent the difference between 28 and the number of ornaments she buys. We want the total number of non-negative integer solutions to the equation:

$$a + b + c + d + e + f + g + h = 28$$

with the following extra constraints: $a \leq 10$, $b \leq 15$, $h \leq 25$.

The total number of solutions without the extra constraints is $\binom{28+8-1}{8-1} = \binom{35}{7}$.

The total number of solutions with $a > 10$ can be determined by "buying" 11 ball ornaments, leaving 17 (at most) to buy. This can be done in $\binom{17+8-1}{8-1} = \binom{24}{7}$ ways.

The total number of solutions with $b > 15$ can be determined by "buying" 16 candy ornaments, leaving 12 (at most) to buy. This can be done in $\binom{12+8-1}{8-1} = \binom{19}{7}$ ways.

The total number of solutions with $h > 25$ can be determined by "not buying" 26 ornaments, leaving 2 (at most) to buy. This can be done in $\binom{2+8-1}{8-1} = \binom{9}{7}$ ways.

The total number of solutions with $a > 10$ AND $b > 15$ (these were double counted previously) can be determined by "buying" 11 ball ornaments and 16 candy ornaments, leaving 1 (at most) to buy. This can be done in $\binom{1+8-1}{8-1} = \binom{8}{7}$ ways.

Notice that none of the other constraints can be simultaneously violated. Using the Inclusion-Exclusion Principle, our final count is:

$$\binom{\mathbf{35}}{\mathbf{7}} - \binom{\mathbf{24}}{\mathbf{7}} - \binom{\mathbf{19}}{\mathbf{7}} - \binom{\mathbf{9}}{\mathbf{7}} + \binom{\mathbf{8}}{\mathbf{7}}$$

Grading: 2 pts total for ${}_{37}C_7$,
4 pts for just ${}_{35}C_7$, 3 pts for restriction on ball ornaments, 3 pts for restriction on candy ornaments, 5 pts for minimum buy, 4 pts for inclusion-exclusion

6) (12 pts) Bob is considering buying raffle tickets for four prizes. The table below summarizes the current information about each of the prizes, right before Bob buys raffle tickets:

Prize	Value	Price for 1 Raffle Ticket	Tickets Already Purchased
Wine	$800	$5	88
Beach	$500	$10	19
Vacation	$1000	$20	17
Dinner	$250	$15	6

Bob has decided that he's going to spend exactly $60 on raffle tickets and that he's only going to buy raffle tickets for a single prize. Also, assume the raffle closes right after Bob makes his purchase. If Bob wants to maximize his expected winnings, which item should he buy raffle tickets for? (Note: No credit is given for the correct answer. All of the credit is given for the justification of the correct answer.) For example, if Bob chooses Beach, then he purchases 6 raffle tickets for it, since each costs $10. After his purchase, there are 25 raffle tickets for the item, of which Bob has 6, so his chance of winning the Beach prize would be 24%. If he chooses this item, what are his expected net winnings? (Note: The net winnings are just his expected prize value minus how much he spent on the raffle tickets to obtain that expected prize value.)

<u>Solution</u>
We compute the expected net winnings of each option, as shown in the chart below:

Prize	Value	Price for 1 Tix	Purchased	# Tix Bob	Tot Tix	P(W)	E(Net Money)
Wine	$800	$5	88	12	100	.12	.12($800) - $60
Beach	$500	$10	19	6	25	.24	.24($500) - $60
Vacation	$1000	$20	17	3	20	.15	.15($1000) - $60
Dinner	$250	$15	6	4	10	.4	.4($250) - $60

Simplifying the last column, we get the following values: $36, $60, $90, and $40. Thus, any choice gives Bob an expected earnings, but the **best choice is buying 3 tickets for the vacation which yields an expected net winnings of $90.**

Grading: 3 pts for each calculation of expected net winnings

7) (12 pts) Let R, S and T be relations such that $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq B \times C$.

(a) Prove or disprove: $(S \cap T) \circ R \subseteq S \circ R \cap T \circ R$
(b) Prove or disprove: $S \circ R \cap T \circ R \subseteq (S \cap T) \circ R$

Solution
(a) This assertion is true. We use direct proof.

Thus, we must show for an arbitrary ordered pair $(a, c) \in (S \cap T) \circ R$, that $(a, c) \in S \circ R \cap T \circ R$.

Given that $(a, c) \in (S \cap T) \circ R$, it follows by definition of relation composition that there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in (S \cap T)$.

By definition of set intersection, we have that $(b, c) \in S$ and $(b, c) \in T$.

Since $(a, b) \in R$ and $(b, c) \in S$, by definition of relation composition $(a, c) \in S \circ R$.
Since $(a, b) \in R$ and $(b, c) \in T$, by definition of relation composition $(a, c) \in T \circ R$.
Thus, by definition of set union, we arrive at the conclusion that $(a, c) \in S \circ R \cap T \circ R$.
This completes the proof of (a).

Grading: 7 pts total
- 2 pts for proof. 0 for disproof.
- 1 pt for stating what to prove
- 2 pts for applying definition of composition to get membership for (a,b), (b,c)
- 2 pts for showing separately that (a,c) is in the two separate compositions

(b) This assertion is false. Consider the following counter-example:

A = {1}, B = {2, 3}, C = {4}

R = {(1, 2), (1, 3)}
S = {(2, 4)}
T = {(3, 4)}

In this example, we have $S \cap T = \emptyset$. Consequently, for this example, $(S \cap T) \circ R = \emptyset$.
In this example, $S \circ R = \{(1,4)\}$ and $T \circ R = \{(1,4)\}$, so $S \circ R \cap T \circ R = \{(1,4)\}$

For the first relation composition, the value b = 2 provides the "bridge" to put (1, 4) in the composition.
For the second relation composition, the value b = 3 provides the "bridge" to put (1, 4) in the composition.
Thus, the given subset relation does not hold for this example as the left hand side, $S \circ R \cap T \circ R$, contains an element, (1, 4), that is not contained on the right-hadn side, $(S \cap T) \circ R$.

Grading: 5 pts total
- 2 for disproof, 0 for proof
- 1 pt for spelling out all sets and relations
- 2 pt if it's valid

8) (14 pts) Let the functions f and g be as follows: f: A → B and g: B → C, with $g \circ f$ being an injective function.

(a) Prove or disprove: f must be an injective function.
(b) Prove or disprove: g must be an injective function.

Solution
(a) This is true. We use direct proof to show that if $g \circ f$ is injective, then *f* is injective. To prove that *f* is injective, we must show that for any *x* and *y* that belong to A, if $f(x) = f(y)$, then $x = y$.

We start with arbitrary *x* and *y* such that $f(x) = f(y)$. Since both values are valid items in the set B, it follows that we can make $f(x)$ and $f(y)$ inputs to the function *g*. Since *g* is a function, we can "take" g of both sides of the equation and maintain equality:

$$g(f(x)) = g(f(y))$$

Now, we will simply invoke the fact that the function $g \circ f$ is injective. By definition, this means that if $g(f(x)) = g(f(y))$, then $x = y$, as desired.

Grading: 8 pts total
- 2 pts for proof, 0 for disproof
- 2 pts for stating that we need to show if f(x) = f(y), then x = y.
- 2 pts for taking g of both sides
- 2 pts for invoking injective property of g(f(x)).

(b) This is false. Consider the following counter-example:

A = {1}
B = {2, 3}
C = {4}
f = {(1, 2)}
g = {(2,4), (3,4)}
$g \circ f$ = {(1,4)}

In this example, *f* is injective and $g \circ f$ is also injective (both only have one ordered pair!). But, the function *g* is NOT injective, since both *g*(2) and *g*(3) are equal to 4. Notice that to create this counter-example, we needed the function f to not be surjective; namely, we had to have one "extra" element in B, that maps to a value in C that the composition function maps in a different way.

Grading: 6 pts total
- 2 pts proof. 0 pts disproof
- 2 pts for spelling out all sets and functions
- 2 pts if counter-example is valid

9) (1 pt) What color is the inside of a pink grapefruit? **Pink** **(Give to All)**