COT 3100 Recitation #11: Final Exam Review - Solutions
11/28/2016-12/2/2016

1) Using mathematical induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Solution

Base case: n = 1, LHS = $\sum_{i=1}^{1} \frac{1}{i(i+1)(i+2)} = \frac{1}{6}$, RHS = $\frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{24} = \frac{1}{6}$. Thus, the statement is true for n = 1.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that:

$$\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} &= \left(\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)}\right) + \left(\frac{1}{(k+1)(k+2)(k+3)}\right) \\
&= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{k(k+3)(k+3)}{4(k+1)(k+2)(k+3)} + \frac{4}{4(k+1)(k+2)(k+3)} \\
&= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)(k+4)}{4(k+2)(k+3)}
\end{aligned}$$

Note: We plugged into the IH on the second line. The factorization was done by guessing that -1 was a root of the cubic function and dividing k + 1 into it to yield $k^2 + 5k + 4$. This was then further factored.

2) There are three people, Adam(A), Belinda(B) and Celine(C), in a room. Each has a hat on and is holding a pet. The colors of the three hats are red(R), orange(O) and purple(P) and the pets are a dog(D), a guinea pig(G) and a hamster(H). Consider the following given information:

1) The person wearing the red hat is not holding the guinea pig.
2) Celine is not holding the hampster.
3) The person wearing the orange hat is holding the hampster.
4) Belinda is holding the dog.

Using any technique, determine which hats each of the individuals is wearing and which pets each of the individuals is holding.

<u>Non-formal Solution</u>

Since Celine is not holding the hamster, and the person with the orange hat IS holding the hamster, Celine's hat isn't orange. In addition, Belinda is holding the dog, which means that her hat isn't orange either, because the person with the orange hat is holding the hamster, not the dog. So, Adam's hat is orange and he's holding the hamster. Since there are two people left to match and one of them must be holding the guinea pig and that person isn't Belinda (since she is holding the dog), it follows that Celine is holding the guinea pig and doesn't have the red hat. Since she doesn't have the red hat, she must have the purple hat. By default, Belinda's hat is red.

Person	Hat Color	Pet
Adam	ORANGE	HAMSTER
Belinda	RED	DOG
Celine	PURPLE	GUINEA PIG

Alternate Formal Solution

The given facts are: 1) $R \rightarrow \bar{G}$, 2) $C \rightarrow \bar{H}$, 3) $O \leftrightarrow H$, 4) $B \leftrightarrow D$. Other given facts are of the form $(\bar{A} \wedge \bar{B}) \rightarrow C$ and will be invoked as necessary.

1) $\rightarrow \bar{H}$, Given
2) $B \rightarrow D$, Given
3) $(B \rightarrow D) \rightarrow (C \rightarrow \bar{D})$, Given since two different people can't have the same pet.
4) $C \rightarrow \bar{D}$, Modus Ponens with 2 and 3.
4) $C \rightarrow (D \vee G \vee H)$, Given, since each person must have a pet.
5) $C \rightarrow G$, Proof by Cases, using 1, 3 and 4
6) $A \rightarrow (D \vee G \vee H)$, Given, since each person must have a pet.
7) $(B \rightarrow D) \rightarrow (A \rightarrow \bar{D})$, Given since two different people can't have the same pet.
8) $A \rightarrow \bar{D}$, Modus Ponens with 2 and 7
9) $(C \rightarrow G) \rightarrow (A \rightarrow \bar{G})$, Given since two different people can't have the same pet.
10) $A \rightarrow \bar{G}$, Modus Pones with 5 and 9
11) $A \rightarrow H$, Proof by Cases, using 6, 8 and 10.
12) $H \rightarrow O$, Given
13) $A \rightarrow O$, Law of Syllogism with 11 and 12.
14) $R \rightarrow \bar{G}$, Given
15) $G \rightarrow \bar{R}$, Contrapositive of 14
16) $C \rightarrow \bar{R}$, Law of syllogism with 5 and 15
17) $(A \rightarrow O) \rightarrow (C \rightarrow \bar{O})$, Given, since two different people can't have the same hat.
18) $C \rightarrow \bar{O}$, Modus Ponens with 13 and 17
19) $C \rightarrow (R \vee P \vee O)$, Given, since each person must have a hat
20) $C \rightarrow P$, Proof by Cases, using 16, 18 and 19.
21) $B \rightarrow (R \vee P \vee O)$, Given, since each person must have a hat
22) $(A \rightarrow O) \rightarrow (B \rightarrow \bar{O})$, Given, since two different people can't have the same hat.
23) $(C \rightarrow P) \rightarrow (B \rightarrow \bar{P})$, Given, since two different people can't have the same hat.
24) $B \rightarrow \bar{O}$, Modus Ponens with 13 and 22
25) $B \rightarrow \bar{P}$, Modus Ponens with 20 and 23
26) $B \rightarrow R$, Proof by cases with 21, 24 and 25.

Note: Proof by cases is a reason that comprises the idea that since each person must have exactly one attribute of three choices, if they don't have two of them, they must have the third.

Adam is wearing the orange hat and is holding the hampster. Belinda is wearing the red had and is holding the dog, and Celine has the purple hat and is holding the guinea pig.

3) Prove or disprove the following assertion about finite sets A and B:

$$P(A) \cup P(B) \subseteq P(A \cup B)$$

Recall that *P(A)* is simply the set of all subsets of A.

Solution

The given assertion is true. We will use direct proof. In order to show that $P(A) \cup P(B)$ is a subset of $P(A \cup B)$, we will take an arbitrary element $X \in P(A) \cup P(B)$ and prove that $X \in P(A \cup B)$.

Given that $X \in P(A) \cup P(B)$, it must be the case that either (a) $X \in P(A)$ or (b) $X \in P(B)$, by definition of union. We must show that the conclusion holds in both of these cases, separately.

If $X \in P(A)$, by definition of power sets, $X \subseteq A$.
By definition of union, it follows that $X \subseteq A \cup B$.
Finally, by definition of power set, it follows that $X \in P(A \cup B)$.

The other case is symmetric to this case:
If $X \in P(B)$, by definition of power sets, $X \subseteq B$.
By definition of union, it follows that $X \subseteq A \cup B$.
Finally, by definition of power set, it follows that $X \in P(A \cup B)$.

4) (15 pts) NTH (Number Theory)

This question will concern all positive integer solutions to the equation $a^2 + b^2 = c^2$ where gcd(a, b) = 1, gcd(a, c) = 1 and gcd(b, c) = 1.

a)(5 pts) Let n be an even integer. Prove that $n^2 \equiv 0 \pmod 4$.

Solution

Since n is even, we can express n = 2a, where a is an integer. Now consider n^2:

$n^2 = (2a)^2 = 4a^2 \equiv 0 \pmod 4$.

b)(5 pts) Let n be an odd integer. Prove that $n^2 \equiv 1 \pmod 4$.

Solution

Since n is odd, we can express n = 2a + 1, where a is an integer. Now consider n^2:

$n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 \equiv 1 \pmod 4$.

c)(5 pts) Prove that for all solutions of $a^2 + b^2 = c^2$ where gcd(a, b) = 1, gcd(a, c) = 1 and gcd(b, c) = 1, that c must be odd and that exactly 1 of a and b must be odd.

Solution

First, consider the situation where a and b are both even. This isn't possible because then gcd(a, b) $\geq$ 2, contradicting the restriction that gcd(a, b) = 1. If one of those values is odd and the other even, then the left hand side is equivalent to $0 + 1 \equiv 1 \pmod 4$. Based on our results, this means that if there's a solution, c must be odd. Finally, consider the situation of both a and b being odd. The left hand side is equivalent to $1 + 1 \equiv 2 \pmod 4$. Based on parts (a) and (b), we know that no perfect squares are equivalent to 2 (mod 4). It follows that there are no solutions to the equation where both a and b are odd. Thus, via an exhaustive search, we've shown that the only categories of solutions that are viable have exactly one of a and b odd and c odd.

5) How many subsets of 5 cards out of 52 regular playing cards contain at least 1 card of each suit?

Solution

We must have a distribution amongst suits of 2, 1, 1, and 1. We can obtain this distribution in 4 ways. So, we can obtain our answer by multiplying by 4 the number of ways to choose 2 clubs, 1 diamond, 1 heart and 1 spade. This can be done in $\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1}$ = 6×13^4 ways. It follows that the total number of subsets that satisfy the restriction is 24×13^4.

6) Each part will concern the situation of rolling three fair six-sided dice. Label these dice A, B and C. Note: Please simplify each of these answers as fractions in lowest terms.

a) (5 pts) What is the probability that the sum of the three dice is 5?

Solution

Here are the ordered triplets (A, B, C) that add up to 5: (1, 1, 3), (1, 3, 1), (3, 1, 1), (2, 2, 1), (2, 1, 2) and (1, 2, 2). There are a total of 6 x 6 x 6 = 216 possible ordered triplets. Thus, the desired probability is $\frac{6}{216} = \mathbf{\frac{1}{6}}$.

b) (5 pts) Given that die A shows a 3, what is the probability that the sum of dice A and B equals what is showing on die C?

Solution

Again, it's easy enough to use brute force and list all valid ordered triples (A, B, C) that satisfy the requirements: (3, 1, 4), (3, 2, 5) and (3, 3, 6). Our sample space is ordered triplets with A = 3. There are 1 x 6 x 6 = 36 such ordered triplets. The desired probability is $\frac{3}{36} = \mathbf{\frac{1}{12}}$.

c) (5 pts) Given that one of the three dice shows a 1 (note that we don't know which one of A, B and C), what is the probability that all three sum to three?

Solution

Only one ordered triplet out of the total 216 sums to three: (1, 1, 1). Thus, to complete the problem, we must determine the total number of ordered triplets that have at least one 1. It's easier for us to calculate the number of ordered triplets without a 1. We do this by noting that we have 5 choices for A, B and C, each for a total of 5 x 5 x 5 = 125 total choices. To get the total number of ordered triplets with a 1, just subtract 125 from the total number of possibilities, 216 to get 91. It follows that the desired probability is $\mathbf{\frac{1}{91}}$.

7) Let $f(x) = 3x^2 - 12x + 7$ with a domain of $x \in (-\infty, 2]$. Determine $f^{-1}(x)$ and its domain and range.

<u>Solution</u>
Switch x and y and solve for y:

$x = 3y^2 - 12y + 7$
$x = 3(y^2 - 4y) + 7$
$x = 3(y^2 - 4y + 4 - 4) + 7$
$x = 3(y^2 - 4y + 4) - 12 + 7$
$x = 3(y-2)^2 - 5$
$x + 5 = 3(y-2)^2$
$\frac{x+5}{3} = (y-2)^2$
$\pm\sqrt{\frac{x+5}{3}} = y - 2$
$y = 2 \pm \sqrt{\frac{x+5}{3}}$

At this point we must determine if the inverse function has a plus sign or a minus sign at the designated location. Given that the original domain was reals less than or equal to 2, that means the range of the inverse function is the same thing. It follows that we must select the negative sign, so that our final answer is:

$$f^{-1}(x) = 2 - \sqrt{\frac{x+5}{3}}$$

The domain of this function is $x \in [-5, \infty)$ and its range, as previous stated, is $(-\infty, 2]$.

8) Recall that a bitwise and (&) between two integers is the result of taking the and of each pair of corresponding bits in the two operands. For example 11 & 13 is 9, because 11 is 1011 in binary, 13 is 1101 in binary, and both of these numbers have 1s in the first and last locations, thus their bitwise and is 1001, which is equal to 9. Define the following relation over **non-zero** 32-bit integers:

Let R = { (a, b) | a and b are 32-bit **non-zero** integers with (a & b) equal to 0 }

Determine, with proof, whether or not R is (a) irreflexive, (b) symmetric and (c) transitive.

Solution

(a) The relation IS irrflexive. For all a that are non-zero 32 bit integers, a & a = a. **(1 pt)** Since a is non-zero, this result isn't zero. Thus, we can conclude for all a for which the relation is over, $(a,a) \notin R$.

(b) The relation IS symmetric. We must show that for, an arbitrary ordered pair (a, b), if $(a, b) \in R$, then $(b, a) \in R$. Consider an arbitrary pair $(a, b) \in R$. We have that a & b = 0. Since the and operator is symmetric itself, it must be the case that b & a = 0. Thus, it follows that $(b, a) \in R$, as desired.

(c) The relation is NOT transitive. Consider the following counter-example: a = 1, b = 2 and c = 1. In binary, these values look like:

a = 01
b = 10
c = 01

In this situation, it's clear to see that a & b = 0, so $(a, b) \in R$, b & c = 0, so $(b, c) \in R$, but a & c = 1, so $(a, c) \notin R$.