

COT 3100 Recitation #9: Probability- Solutions
11/7-10/2016

Warm-Up Problems

1) A class collects \$50 to buy flowers for a classmate who is in the hospital. Roses cost \$3 each and carnations cost \$2 each. No other flowers are to be used. How many different bouquets could be purchased for exactly \$50?

Solution

We want to know the number of non-negative integer solutions to the equation $3r + 2c = 50$, where r is the number of roses bought and c is the number of carnations bought. If we consider the equation mod 3, it reduces to $2c \equiv 2 \pmod{3}$. The solution to this equation is $c \equiv 1 \pmod{3}$. (Formally we solve it by multiplying the equation through by $2^{-1} \pmod{3}$ but it's easy enough to eyeball this solution.) So, valid values of c are $c = 1, 4, 7, 10, 13, 16, 19, 22$ and 25 . Note that we include the last value since there is no restriction that says that at least one rose must be bought. Thus, there are **9** possible bouquets that can be bought.

2) For each positive integer n , the mean of the first n terms of a sequence is n . What is the 2008th term of the sequence?

Solution

Let a_n be the n^{th} term in the sequence and S_n be the sum of the first n terms of the sequence. Since the average of these first n terms is n , it follows that this sum is n^2 . We can solve for the 2008th term as follows:

$$a_{2008} = S_{2008} - S_{2007} = 2008^2 - 2007^2 = (2008 - 2007)(2008 + 2007) = \mathbf{4015}$$

3) A circle has radius of $\log_{10}(a^2)$ and a circumference of $\log_{10}(b^4)$. What is $\log_a b$?

Solution

We know that the circumference of a circle is $2\pi r$, where r is the radius of that circle. Thus the circumference of the circle described above is $2\pi \log_{10}(a^2) = \log_{10}(a^{4\pi})$. Setting the two expressions for the circumference of the circle equal to each other we get:

$$\begin{aligned} \log_{10}(a^{4\pi}) &= \log_{10}(b^4) \\ a^{4\pi} &= b^4 \\ \log_a a^{4\pi} &= \log_a b^4 \\ 4\pi &= 4\log_a b \\ \log_a b &= \pi \end{aligned}$$

4) A rectangular floor measures a feet by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the entire floor. How many possibilities are there for the ordered pair (a, b) ?

Solution

The painted area is $(a - 2)$ feet by $(b - 2)$ feet, so we have the following equation:

$$\begin{aligned}2(a - 2)(b - 2) &= ab \\2ab - 4a - 4b + 8 &= ab \\ab - 4a - 4b + 8 &= 0 \\ab - 4a - 4b + 16 &= 8 \\(a - 4)(b - 4) &= 8\end{aligned}$$

Since a and b are positive integers, we can quickly reason out that both terms in the product above are positive as well. It follows that $a - 4 = 1$, or 2 , meaning that $a = 5$ or 6 . The corresponding values for b are $b = 12$ or 8 . The corresponding ordered pairs are $(5, 12)$ or $(6, 8)$. Thus, there are **two** possibilities for the ordered pair (a, b) .

5) Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

Solution

Let Brenda's normal rate of work be laying x bricks per hour and Brandon's be laying y bricks per hour. It follows that the chimney has $9x$ bricks, which is the same as $10y$ bricks. Their collective rate of work is $x + y - 10$ bricks per hour. This yields the equation:

$$\begin{aligned}5(x + y - 10) &= 10y \\x + y - 10 &= 2y \\x &= y + 10\end{aligned}$$

Now, substitute into our other equation:

$$\begin{aligned}9x &= 10y \\9(y + 10) &= 10y \\9y + 90 &= 10y \\y &= 90\end{aligned}$$

It follows that there are $10 \times 90 = \mathbf{900}$ bricks in the chimney.

Probability Problems

6) Disease A occurs in 0.05% of the population. If a person does NOT have the disease and takes a test for the disease, the test correctly indicates that they don't have the disease 97% of the time. If a person has the disease and takes the same test, the test correctly indicates that they do have the disease 99% of the time. Given that you've taken the test and have tested positive for disease A, what is the probability you actually have disease A? Given that you've taken the test and have tested negative for disease A, what is the chance that you have the disease anyway?

Solution

Let X be the event you have disease A and Y be the event that you test positive for it.

$$P(Y) = P(X)P(Y|X) + P(\bar{X})P(Y|\bar{X})$$
$$P(Y) = (.0005)(.99) + (.9995)(.03) = .03048$$

Thus, 3.048% of the population *tests* positive for the disease. The probability of having the disease AND testing positive for it are $P(X \cap Y) = P(X)P(Y|X) = (.0005)(.99) = .000495$.

Thus, the desired conditional probability, $P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{.000495}{.03048} \sim .016 = \mathbf{1.6\%}$

To find the other probability, the chance we have the disease given that we tested negative for it:

$$P(X|\bar{Y}) = \frac{P(X \cap \bar{Y})}{P(\bar{Y})} = \frac{P(X)P(X|\bar{Y})}{P(X)P(X|\bar{Y}) + P(\bar{X})P(\bar{X}|\bar{Y})} = \frac{(.0005)(.01)}{(.0005)(.01) + (.9995)(.97)} \sim \mathbf{5.16 \times 10^{-6}}$$

Thus, it's far less likely for there to be a mistake where a person has the disease and is told that they don't have it than the other way around.

7) Anderson gets 75% of the multiple choice questions in COT 3100 he answers. Given a set of 20 questions, what is the chance that he gets at least 18 of them? Write your answer using combinations and then use a calculator to get a decimal approximation for it.

Solution

Using the binomial distribution, we work out the probability that Anderson gets 18, 19 or 20 questions correct:

$$\binom{20}{18} (.75)^{18} (.25)^2 + \binom{20}{19} (.75)^{19} (.25) + (.75)^{20} \sim \mathbf{9.13\%}$$

8) If A and B are events and $p(A) = 3/7$, $p(A \cap B) = 1/5$, $p(A | B) = 1/2$ calculate $p(B)$, $p(B|A)$ and $p(B | \bar{A})$, are A and B independent? Mutually exclusive?

Solution

$$\frac{1}{2} = p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{\frac{1}{5}}{p(B)}$$

Solving the equation above for $p(B)$, we find that $p(B) = \frac{2}{5}$.

$$p(B|A) = \frac{p(B \cap A)}{p(A)} = \frac{\frac{1}{5}}{\frac{3}{7}} = \frac{7}{15}$$

$$p(B|\bar{A}) = \frac{p(B \cap \bar{A})}{p(\bar{A})} = \frac{p(B)p(\bar{A}|B)}{\frac{4}{7}} = \frac{\frac{2}{5} \times \frac{1}{2}}{\frac{4}{7}} = \frac{7}{20}$$

The last equation needs some explaining. First note that $p(\bar{A}) = 1 - p(A)$. Also, note that, $p(\bar{A}|B) = 1 - p(A|B)$.

If A and B were independent events, then $p(A|B) = p(A)$. But, the given information clearly shows that these two values are not equal to one another. Thus, we can conclude that events A and B are not independent. In addition, the events, by definition aren't mutually exclusive since $p(A \cap B)$ is non-zero.

9) A bag of popping corn contains $2/5$ white kernels and $3/5$ yellow kernels. Only $1/3$ of the white kernels will pop, whereas $1/2$ of the yellow ones will pop. A kernel is selected at random from the bag, and pops when placed in the popper. What is the probability that the kernel selected was white?

Solution

Let A be the event that a white kernel is selected from the bag and let B be the event that a kernel selected pops. Note that event \bar{A} , denotes the event that a yellow kernel is selected since there are only two types of kernels.

$$p(B) = p(A)p(B|A) + p(\bar{A})p(B|\bar{A})$$

$$p(B) = \frac{2}{5} \times \frac{1}{3} + \frac{3}{5} \times \frac{1}{2} = \frac{2}{15} + \frac{3}{10} = \frac{4 + 9}{30} = \frac{13}{30}$$

The questions asks us to find $p(A|B)$:

$$p(A|B) = \frac{p(A)p(B|A)}{p(B)} = \frac{\frac{2}{5} \times \frac{1}{3}}{\frac{13}{30}} = \frac{\frac{2}{15}}{\frac{13}{30}} = \frac{4}{13}$$

10) Bob has been chosen for a half-time promotion at a local basketball game. He will take as many three point shots as he can in 30 seconds. If he makes at least one of the shots, he wins \$1000. After some practice, Bob knows that he'll get off 5 shots in the given time and that he chance of making an single shot is 15%. What is the expected value of Bob's prize for this promotion?

Solution

It's easier to calculate his probability of not making a shot than calculating his probability of making at least one shot. Since he makes .15 of his shots, his chance of missing 5 shots in a row is simply $(.85)^5 \sim .44471$. It follows that the chance he makes at least one shot is roughly .55629. Thus, it follows that his expected winnings are $.55629 \times \$1000 = \mathbf{\$556.29}$.

11) Mikey is taking a matching quiz with 4 items on it. Unfortunately, Mikey didn't study and he completely guesses the answers, knowing that each of the four words will match to a different definition of the four given. What is the chance that Mikey gets all four definitions incorrect? What is the chance that he gets precisely 3 of the definitions correct? (We assume he matches each word to a unique definition listed.)

Solution

There are 24 possible answers Mikey can give corresponding to the 4! orderings of the definitions he can give. Let's count how many of them correspond to Mikey missing all 4 questions. Without loss of generality, assume that the correct answers are just 1, 2, 3 and 4. Here are the permutations of answers Mikey can give that don't match any of those answers, listed in lexicographical order:

- 2, 1, 4, 3
- 2, 3, 4, 1
- 2, 4, 1, 3
- 3, 1, 4, 2
- 3, 4, 1, 2
- 3, 4, 2, 1
- 4, 1, 2, 3
- 4, 3, 1, 2
- 4, 3, 2, 1

Thus, Mikey's chance of missing all 4 questions is $\frac{9}{24} = \frac{3}{8}$.

It's impossible for Mikey to get 3 questions correct on the matching quiz. Thus the probability this occurring is **0**. If three of them were to match, by default the fourth answer would also. Thus, every permutation (possible answer given the rules) corresponds to either 0 correct, 1 correct, 2 correct or 4 correct.