COT 3100 Recitation #8: Counting Solutions
10/24-28/2016

Warm-Up Problems

1) Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?

Solution

Let the ages of the three kids be a, b and c, respectively. The given information yields:

T = a + b + c
T - N = 2(a - N + b - N + c - N)

Simplifying the second equation and substituting for a + b + c in it we get:

T - N = 2(a + b + c - 3N)
T - N = 2(T - 3N)
T - N = 2T - 6N
T = 5N
T/N = **5**

2) The geometric series a + ar + ar^2 + … has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is a + r?

Solution

Using the formula for the sum of a geometric sequence, we have $\frac{a}{1-r} = 7$. The terms involving the odd powers of r ALSO form a geometric sequence with first term ar and common ratio r^2. This yields the equation $\frac{ar}{1-r^2} = 3$. We can solve this system as follows:

$\frac{ar}{(1-r)(1+r)} = 3$
$\frac{7r}{(1+r)} = 3$, by substituting for $\frac{a}{1-r}$ from the original equation.
$7r = 3(1+r)$
$7r = 3 + 3r$
$4r = 3$
$r = \frac{3}{4}$

Now, we can substitute r into the original equation to get $\frac{a}{1-\frac{3}{4}} = 7$, $\frac{a}{\frac{1}{4}} = 7$, $4a = 7$, $a = \frac{7}{4}$.

It follows that $a + r = \frac{3}{4} + \frac{7}{4} = \mathbf{\frac{5}{2}}$.

3) A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each junior receive on the test?

<u>Solution</u>

Let there be 9x seniors and x juniors in the class. (This encodes the information that the class is 90% seniors.) Let j be the score all of the juniors received. The sum of the scores of the seniors is 83(9x). The sum of the scores the juniors received is jx, so the sum of all the scores is 83(9x) + jx and the number of students taking the exam was 10x, where x is a positive integer. We can use this information to solve for j:

$$\frac{83(9x) + jx}{10x} = 84$$
$$(747 + j)x = 840x$$
$$747 + j = 840$$
$$j = \mathbf{93}$$

Note: We can divide by x in the third step since x isn't 0.

4) If a is a non-zero integer and b is a positive number such that $ab^2 = \log_{10}b$, what is the median of the set {0, 1, a, b, 1/b}?

<u>Solution</u>

We can solve for a and get $a = \frac{log_{10}b}{b^2}$. Note that b can not be 1. If we set b = 1, then the numerator of the fraction is 0, setting a to 0, which isn't allowed since we're given that a is non-zero. b can not be greater than 1. If it is, then the numerator is a positive value and the denominator is a positive value greater than the numerator (the function $f(x) = x^2$ is strictly greater than the function $g(x) = \log_{10}x$ for all x > 1.), which would set a to a real number greater than 0 and less than 1, which also isn't allowed.

It follows that b is less than 1 and $\log_{10}b$ is negative. An example of a possible solution is b = $\frac{1}{10}$. In this case, a = -100. In fact, any solution must have $\log_{10}b$ be an integer. We can answer the question by noting that 0 < b < 1, 1 < 1/b < ∞, which implies that a < 0 because the restriction on b makes the numerator of the fraction above negative and the denominator positive. It follows that the ordering of the values given is: a, 0, b, 1, 1/b.

Thus, the median value of the list is **<u>b.</u>**

5) The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number whose digits read the same forwards and backwards. For example, 11011 and 1221 are both palindromes in base 3.)

Solution

First calculate 2007 in base 3:

3 | 2007
3 | 669 R 0
3 | 223 R 0
3 | 74 R 1
3 | 24 R 2
3 | 8 R 0
3 | 2 R 2
3 | 0 R 2

Notice that this has 7 digits, so we must count all palindromes of lengths 1, 2, 3, 4, 5 and 6 digits in base 3. There are 2, 2, 6, 6, 18 and 18 of these respectively, since for the leading digit we always have 2 choices and then we have 3 choices for following digits until we get to the "middle of the number." This is a sum of 52 palindromes of 6 of fewer digits in base 3.

Thus, 2007 = 2202100_3. So, what we want is the number of strings of length 7 of 0s 1s and 2s, starting with a 1 or 2 that have a numerical value less than or equal to what's listed above and are palindromes. Notice that if we fix the first digit to 1, we are completely free to pick the 2nd, 3rd and 4th digits. These correspond to 1 x 3 x 3 x 3 = 27 choices, using the multiplication principle.

If we pick 2 for the first digit, then we can choose 0 or 1 for the 2nd digit and we have 3 choices for the 3rd and 4th digits, leading to 2 x 3 x 3 = 18 choices (corresponding to strings that start 20 or 21.)

For strings that start 22, the third digit must be 0 to be less than or equal to the given number. Finally, if we fix 220 for the first three digits, all three choices for the fourth digit lead to a valid palindrome to count: 2200022, 2201022 and 2202022, all of which are less than 2202100.

Our final count is 52 + 27 + 18 + 3 = **<u>100</u>**.

Counting Problems

6) How many permutations (of length 11) are there of the letters in the word MATHEMATICS?

Solution

We want the number of permutations where some letters are repeated. Namely, we have 2 Ms, 2 As, and 2 Ts, with a total of 11 letters. Plugging into the permutation formula, we get $\frac{\mathbf{11}!}{\mathbf{2!2!2!}}$.

7) A string, $c_1c_2c_3 \ldots c_n$, is called an ascending string if c_i comes strictly before c_{i+1}, alphabetically, for all $1 \leq i \leq n-1$. How many ascending strings of length 4 can be formed? (For example, ACMQ and FRSZ are both ascending strings of length 4 but BQQZ and BRTJ are not.)

Solution

Notice that for each ***choice*** of 4 letters out of 26, we can form precisely one ascending string. For example, if the letters Q, M, A and R were chosen, the one corresponding ascending string is AMQR. Similarly, each ascending string maps to precisely one choice of 4 letters out of 26. It follows that to count the number of ascending strings we can simply count the number of combinations of 4 letters out of 26, which is $\binom{\mathbf{26}}{\mathbf{4}}$.

8) How many integers in between 1 and 1000 are divisible by 2, 3 or 5?

Solution

Let set A be the set of integers in the range divisible by 2, the set B be the set of integers in the range divisible by 3 and the set C be the set of integers in the range divisible by 5. Using the inclusion exclusion principle we find that:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Given two integers x and y such that gcd(x, y) = 1, we know that in order for a value to be divisible by x and y, it must be divisible by xy. Furthermore, in the set of integers from 1 to n, inclusive, the number of values divisible by x is $\left\lfloor \frac{n}{x} \right\rfloor$, because in every group of x consecutive integers, starting with x = 1, the last value is divisble by x. It follows that:

$$|A \cup B \cup C| = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$
$$|A \cup B \cup C| = 500 + 333 + 200 - 166 - 100 - 66 + 33 = \mathbf{734}$$

9) How many permutations of “ATTTCCCAAGGG” are there such that all “C”s and “G”s appear together consecutively? (Note: The C's and G's can be in any order so, for example, the string TACGCGCGTTAA should be counted even though the three Cs aren't right next to each other.)

Solution

Put together the three Cs and 3Gs together into one "superletter" so that we have 7 total letters: 3 As, 3Ts and "CCCGGG". Thus, we are permuting 7 letters with two letters repeated three times. We can do this in $\frac{7!}{3!3!}$ ways. Unfortunately, this count only accounts for 3 Cs followed by 3 Gs and no other arrangement of consecutive Cs and Gs. There are $\frac{6!}{3!3!}$ orderings of Cs and Gs that are stuck together. For each of the old permutations described, we can create $\frac{6!}{3!3!}$ ones by simply rearranging the Cs and Gs (without touching the locations of the As and Ts). It follows that the total number of arrangements of strings where the Cs and Gs are all next to each other is $\frac{\mathbf{7!6!}}{\mathbf{3!3!3!3!}} = 2800$.

10) How many permutations of 4 As, 5 Bs and 6 Cs do not contain two consecutive C's in a row?

Solution

Let's use the As and Bs as separators. For the time being let's put all the As before the Bs yielding the following arrangement:

__ A __ A __ A __ A __ B __ B __ B __ B __ B __

We must choose 6 of the 10 slots (labeled with the underscores) to place the Cs. This can be done in $\binom{10}{6}$ ways. For each of these choices, we can rearrange the As and Bs in $\frac{9!}{4!5!}$ ways, using the permutation formula (with repeated letters). It follows that the total number of permutations we seek is $\binom{\mathbf{10}}{\mathbf{6}}\frac{\mathbf{9!}}{\mathbf{4!5!}} = \binom{10}{6}\binom{9}{5}$.