COT 3100 Recitation #7: Exam #2 Review - Induction Solutions
10/17-10/21/2016

1) Use induction on n to prove that $4^{2n} - 15n - 1$ is divisible by 225 for all non-negative integers n.

Solution

Base case: n=0. The expression evaluates to $4^{2(0)} - 15(0) - 1 = 0$, which is divisible by 225.

Inductive hypothesis: Assume for an arbitrary integer n=k that $4^{2k} - 15k - 1$ is divisible by 225. Namely, assume that $4^{2k} - 15k - 1 = 225a$ for some integer a.

Inductive step: Prove for n=k+1 that $4^{2(k+1)} - 15(k+1) - 1$ is divisible by 225. Namely, show that $4^{2(k+1)} - 15(k+1) - 1 = 225b$, for some integer b.

$4^{2(k+1)} - 15(k+1) - 1 = 4^{2k+2} - 15k - 15 - 1$
$= 4^2 4^{2k} - 15k - 16$
$= 16(4^{2k}) - 15k - 16$
$= 16(4^{2k}) - (16)15k - 16 + 15(15k)$, add/subtracting 15(15k)
$= 16(4^{2k} - 15k - 1) + 225$, factoring out 16 from the first 3 terms
$= 16(225a) + 225$, using the inductive hypothesis
$= 225(16a+1)$,

Since a is an integer, let b = 16a + 1, and we are done. Thus, for all non negative integers n, $4^{2n} - 15n - 1$ is divisible by 225.

2) Using induction on n, prove that the following formula is true for all positive integers n.

$$\sum_{i=1}^{n} \frac{i(i+1)(i+2)}{6} = \frac{n(n+1)(n+2)(n+3)}{24}$$

Solution

Base case: n = 1 LHS = $\sum_{i=1}^{1} \frac{i(i+1)(i+2)}{6} = \frac{1(2)(3)}{6} = 1$

RHS = $\frac{1(1+1)(1+2)(1+3)}{24} = \frac{24}{24} = 1$

Inductive hypothesis: Assume for an arbitrary integer n=k that

$$\sum_{i=1}^{k} \frac{i(i+1)(i+2)}{6} = \frac{k(k+1)(k+2)(k+3)}{24}$$

Inductive step: Prove for n=k+1 that $\sum_{i=1}^{k+1}\frac{i(i+1)(i+2)}{6}=\frac{(k+1)(k+2)(k+3)(k+4)}{24}$.

$\sum_{i=1}^{k+1}\frac{i(i+1)(i+2)}{6}=\sum_{i=1}^{k}\frac{i(i+1)(i+2)}{6}+\frac{(k+1)(k+2)(k+3)}{6}$ *(2 pts)*

$=\frac{k(k+1)(k+2)(k+3)}{24}+\frac{(k+1)(k+2)(k+3)}{6}$, using the IH.

$=\frac{k(k+1)(k+2)(k+3)}{24}+\frac{4(k+1)(k+2)(k+3)}{24}$, getting com. den.

$=\frac{(k+1)(k+2)(k+3)}{24}(k+4)$, factoring out common terms

$=\frac{(k+1)(k+2)(k+3)(k+4)}{24}$, as desired.

3) Use induction to show that $\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^n=\begin{pmatrix}1 & 0\\ -2^n+1 & 2^n\end{pmatrix}$ for all positive integers n.

<u>Solution</u>

Base case: n=1 LHS = $\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^1=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$, RHS = $\begin{pmatrix}1 & 0\\ -2^1+1 & 2^1\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$,

so the base case is true.

Inductive hypothesis: Assume for an arbitrary integer n=k that $\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^k=\begin{pmatrix}1 & 0\\ -2^k+1 & 2^k\end{pmatrix}$.

Inductive step: Prove for n=k+1 that $\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 0\\ -2^{k+1}+1 & 2^{k+1}\end{pmatrix}$

$\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}^k\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\ -2^k+1 & 2^k\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$, using the inductive hypothesis

$=\begin{pmatrix}1 & 0\\ -2^k+1-2^k & 2(2^k)\end{pmatrix}$, multiplying out the matrices

$=\begin{pmatrix}1 & 0\\ -2^{k+1}+1 & 2^{k+1}\end{pmatrix}$, using exponent rules for simplification.

Thus, it follows that the assertion is true for all positive integers n.

4) The n[th] Harmonic number, denoted H_n is defined as follows:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} \frac{i}{i+1} = (n+1) - H_{n+1}$$

Solution

Base case: n=1 LHS = $\sum_{i=1}^{1} \frac{i}{i+1} = \frac{1}{1+1} = \frac{1}{2}$

RHS = $(1+1) - H_{1=1} = 2 - (1 + \frac{1}{2}) = 2 - \frac{3}{2} = \frac{1}{2}$

Thus, the equation holds for n=1.

Inductive hypothesis: Assume for an arbitrary n=k, that

$$\sum_{i=1}^{k} \frac{i}{i+1} = (k+1) - H_{k+1}$$

Inductive step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{i}{i+1} = ((k+1)+1) - H_{(k+1)+1} = (k+2) - H_{k+2}$$

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{i}{i+1} &= (\sum_{i=1}^{k} \frac{i}{i+1}) + \frac{k+1}{k+2} \\ &= (k+1) - H_{k+1} + \frac{k+1}{k+2} \\ &= (k+1) - H_{k+1} + \frac{k+2-1}{k+2} \\ &= (k+1) - H_{k+1} + \frac{k+2}{k+2} - \frac{1}{k+2} \\ &= (k+1) - H_{k+1} + 1 - \frac{1}{k+2} \\ &= (k+2) - (H_{k+1} + \frac{1}{k+2})\end{aligned}$$

$= (k+2) - H_{k+2}$, proving the formula true for all positive integers n.

5) Use induction on n to prove the following inequality for all positive integers n:

$$\sum_{i=0}^{n} 3^i < \frac{3^{n+1}}{2}$$

Solution

Base case: n=1 LHS = $\sum_{i=0}^{1} 3^i = 3^0 + 3^1 = 4$ RHS = $3^{1+1}/2 = 4.5$

So, LHS < RHS and the base case holds for n=1.

Inductive hypothesis: Assume for an arbitrary integer n=k that

$$\sum_{i=0}^{k} 3^i < \frac{3^{k+1}}{2}$$

Inductive Step: Prove for n=k+1 that

$$\sum_{i=0}^{k+1} 3^i < \frac{3^{k+1+1}}{2}$$

$$\sum_{i=0}^{k+1} 3^i = \sum_{i=0}^{k} 3^i + 3^{k+1}$$

$$< \frac{3^{k+1}}{2} + 3^{k+1}, \text{ using the inductive hypothesis}$$

$$= 3^{k+1}(\frac{1}{2} + 1)$$

$$= 3^{k+1}(\frac{3}{2})$$

$$= 3^{k+1}(\frac{3}{2})$$

$$= \frac{3^{k+2}}{2}$$

6) The Fibonacci numbers are defined as follows: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all integers n > 1. Prove the following formula for all positive integers n:

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$$

<u>Solution</u>

Base case: n=1 LHS = $\sum_{i=1}^{1} \frac{F_{i-1}}{2^i} = \frac{F_{1-1}}{2^1} = 0$ RHS = 1 - $F_{1+2}/2^1$ = 1 - 2/2 = 0

Inductive Hypothesis: Assume for an arbitrary integer n=k that

$$\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$$

Inductive Step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+3}}{2^{k+1}}$$

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + -\frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{2F_{k+2}}{2^{k+1}} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{(2F_{k+2} - F_k)}{2^{k+1}}$$

$$= 1 - \frac{(F_{k+2} + (F_{k+2} - F_k))}{2^{k+1}}$$

$$= 1 - \frac{(F_{k+2} + F_{k+1})}{2^{k+1}}$$

$$= 1 - \frac{F_{k+3}}{2^{k+1}}$$