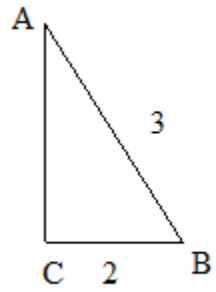
COT 3100 Recitation #7: Counting (Spring 2017) Solutions
3/20-24/2017

Warm-Up Problems

1) Triangle ABC has a right angle at C. If sin A = 2/3, what is tan B?

Solution
Here is a picture of the triangle:



Using the Pythagorean Theorem, the missing side is $\sqrt{5}$, since 3^2 - 2^2 = 5. It follows that $\tan B = \frac{\sqrt{5}}{2}$. Note: It's not know that the side length of BC is 2, rather it's 2x and AB is 3x, for some positive real number x. But, for questions of this nature, only the ratio of the sides matter, so we can correctly solve the problem by finding any triangle that satisfies the given information.

2) In a certain population the ratio of the number of women to the number of men is 11 to 10. If the average age of the women is 34 and the average age of the men is 32, then what is the average age of the population?

Solution
Let 11x equal the number of women and 10x equal the number of men. There are 21x people in the population. If the average age of the women is 34, the sum of their ages is 34(11x) = 374x. If the average age of the men is 32, the sum of their ages is 32(10x) = 320x. The sum of everyone's ages is 374x + 320x = 694x. Thus, the average age of the population is $\frac{694x}{21x} = 33\frac{1}{21}$.

3) If $log_7\big(log_3(log_2 x)\big) = 0$, what is $x^{-0.5}$?

Solution
Get rid of the first log to yield: $\big(log_3(log_2 x)\big) = 7^0 = 1$. Get rid of the second log to get: $log_2 x = 3^1$. Finally, we find that $x = 2^3 = 8$. It follows that $x^{-0.5} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$.

4) What is the units digit of $3^{1001}7^{1002}13^{1003}$?

Solution
The units digit of 3 to successive powers cycles 1, 3, 9, 7.
The units digit of 7 to successive powers cycles 1, 7, 9, 3.
Thus, 3^{1001} ends in 3, 7^{1002} ends in 9, and 13^{1003} ends in 7.
Multiplying 3 x 9 x 7 = 27 x 7 = 189, which ends in 9.

5) Let x = .123456789101112…998999, where the digits are obtained by writing the integers 1 through 998 in order. What is the 2017^{th} digit to the right of the decimal point?

Solution
1 - 9 uses 9 digits
10 - 99 uses 2 x 90 = 180 digits
d00 - d99 uses 3 x 100 = 300 digits, for each d = 1, 2, …, 9

2017 - 189 = 1828.

Thus, we can zip through 1828/300 = 6 (using integer division) ranges of integers. Thus, after we've counted "699", we have gone through 189 + 1800 = 1989 digits. We must then read through 2017 - 1989 = 28 more digits. We read through 700 - 708, which is 27 more digits. The following digit, the 2017^{th} one, is 7.

Just for kicks, here's a slow Java program to check this result:

```
import java.util.*;
public class digits {
      public static void main(String[] args) {
            String res = "";
            for (int i=1; i<=999; i++)
                  res = res + i;
            System.out.println(res.charAt(2016));
      }
}
```

Counting Problems

6) Bob has the following tasks to complete for the morning: brush teeth, eat breakfast, pay bills, do homework, call his friend Sue. If he must brush his teeth before eating breakfast and there are no other restrictions for the ordering of his tasks, in how many ways can he complete the five tasks?

Solution
There are 5! = 120 total orderings of the tasks. Of these, half of them have brushing teeth preceding eating breakfast while the other half don't. So, the result is 60. Another way to look at it is that we can place the other 3 items in 5 x 4 x 3 = 60 of the ordered slots. Once we place these 3 items in any of the slots 1 - 5, we are forced to put brushing teeth in the first empty slot and eating breakfast in the second empty slot.

7) How many strings of 10 letters can be formed such that no substring of 3 letters within the string contain a repeated letter? (For example, ABCABCABCY should be counted since none of the substrings ABC, BCA, CAB, ABC, BCA, CAB, ABC or BCY have any repeated letters, but ABCDEFGHIH should not be counted because the substring HIH contains H twice.)

Solution
We have 26 choices for the first letter, 25 choices for the second letter and 24 choices for the third letter. From there, for all subsequent letters, precisely 2 choices (the two previous letters) are not allowed, so there are always 24 choices for letters 3 through 10. Using the multiplication principle, the total number of strings we desire is $26 \times 25 \times 24^8$.

8) A sorted string is a string which has its letters in alphabetical order. How many sorted strings of 5 letters can be formed from the letters {A, B, C, D, E, F, G, H, I, J}?

Solution
Each choice of 5 letters out of the 10 listed will produce a single valid string since once we know which letters we're using, we can only place them in one order. Thus, there are $\binom{10}{5}$ such sorted strings.

9) Each student (and the students are distinguishable) in a class of 10 selects one of 5 pizzas (cheese, pepperoni, mushroom, vegetable, supreme). In how ways can the students order such that at least one student orders a cheese pizza?

Solution
Let's answer the opposite question: In how many ways can the students order pizzas without getting a cheese pizza. In this case each student only has 4 choices, so they can choose their pizzas in 4^{10} ways, using the multiplication principle. Subtract this from the total number of ways to choose pizzas, 5^{10} to yield $5^{10} - 4^{10}$ ways in which at least one student gets a cheese pizza.

10) Each student (and the students are distinguishable) in a class of 10 selects one of 5 pizzas (cheese, pepperoni, mushroom, vegetable, supreme). In how ways can the students order such that at least one student orders a cheese pizza and another orders a pepperoni pizza?

Solution

We can initially follow the solution idea for before, yielding $5^{10} - 4^{10} - 4^{10}$, where we've subtracted out the ways in which no cheese pizza is selected and the ways in which no pepperoni pizza is selected. But, in subtracting these two things out, we subtracted out some combinations that have neither cheese nor pepperoni pizzas twice. In particular, all of the orders taken completely from {mushroom, vegetable, supreme} have been subtracted out twice. To correct for this, we add these 3^{10} orders back in to get a final tally of $5^{10} - 2(4^{10}) + 3^{10}$.