COT 3100 Recitation #6: Sums and Matrix Practice Solutions
10/10-14/2016

Warm-Up Problems

1) Mary is 20% older than Sally and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?

Solution

Let S be Sally's age. Then Mary is 1.2S years old and Danielle is $\frac{S}{.6}$ years old. Adding the ages we get the equation:

$$1.2S + S + \frac{S}{.6} = 23.2$$
$$\frac{6}{5}S + S + \frac{5}{3}S = 23.2$$
$$\frac{18}{15}S + S + \frac{25}{15}S = 23.2$$
$$\frac{58}{15}S = 23.2$$
$$S = 23.2 \times \frac{15}{58} = 15 \times .4 = 6$$

Thus, Mary is $1.2 \times 6 = 7.2$ years old. She will be **8** on her next birthday.

2) For how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

Solution

Under the large radical we need a perfect square to obtain an integer overall. Thus, we must have either 0, 1, 4, 9, 16, 25, 36, 49. 64, 81 or 100 under the large radical. Each of these values is possible to obtain. For example, if we want 9 under the large radical, then we want $\sqrt{x}$=111, which occurs when we set $x = 111^2$. Thus, there are **11** real values for which the given expression is an integer.

3) Two farmers agree that pigs are worth $300 and that goats are worth $210. When one farmer owes the other money, he pays the debt in pigs and goats, with "change" received in the form of goats or pigs, as necessary. What is the amount of the smallest positive debt that can be resolved this way?

Solution

The gcd(210, 300) = **30.** (You can run Euclid's Algorithm to verify this.) It's relatively easy to see that only positive debts divisible by 30 can be paid off exactly (this was covered in class in detail). The Extended Euclidean algorithm shows that you can always find a solution to the equation 210x + 300y = gcd(210, 300) in integers x and y. In this problem, one integer would indicate how many of one animal one farmer paid and the negative of the other would indicate how many of the other animal were received.

4) The function f has the property that for each real number x in its domain, 1/x is also in its domain and $f(x) + f\left(\frac{1}{x}\right) = x$. What's the largest set of real numbers that can be the domain of f?

Solution

Now, consider plugging in an arbitrary value of x along with $\frac{1}{x}$ into the given equation that the function satisfies. We find that $f(x) + f\left(\frac{1}{x}\right) = x$ and $f\left(\frac{1}{x}\right) + f(x) = \frac{1}{x}$. The logical conclusion here is that $x = \frac{1}{x}$, since the left hand side of both equations are the same. The only values of x that satisfy this equation are $x = \pm 1$. It follows that the largest possible domain the function could have is the set **{-1, 1}.**

5) Let A, B and C be the centers of three mutually tangent circles such that the distance between A and B is 3, the distance between A and C is 4 and the distance between B and C is 5. What is the sum of the areas of the three circles? (Hint: the sum of the radii of the circles with centers A and B is 3, and a similar deduction is true about the sum of the radii of the other two pairs of circles.)

Solution

Let the three radii of the circles be r_A, r_B, and r_C. Using the given information, we set up the following three equations:

$$r_A + r_B = 3$$
$$r_A + r_C = 4$$
$$r_B + r_C = 5$$

Adding all three equations, we get: $2(r_A + r_B + r_C) = 12$. Thus, the sum of the three radii is 6. Finally, we can solve for all three radii separately by subtracting each of the equations above from the equation : $r_A + r_B + r_C = 6$. This yields $r_A = 1$, $r_B = 2$ and $r_C = 3$. It follows that the sum of the areas of the three circles is $\pi(1^2) + \pi(2^2) + \pi(3^2) = \mathbf{14\pi}$**.**

Summation and Matrix Problems

6) Determine $\sum_{i=1}^{2n}(3i+5)$, in terms of n.

Solution

$$\sum_{i=1}^{2n}(3i+5) = \left(3\sum_{i=1}^{2n} i\right) + \sum_{i=1}^{2n} 5$$
$$= \frac{3(2n)(2n+1)}{2} + 5(2n)$$
$$= 3n(2n+1) + 5(2n)$$
$$= 6n^2 + 3n + 10n$$
$$= \mathbf{6n^2 + 13n}$$

7) Determine $\sum_{i=n+1}^{2n}((2i+1)(i-3))$.

Solution

$\sum_{i=n+1}^{2n}((2i+1)(i-3)) =$

$$\sum_{i=n+1}^{2n}(2i^2-5i-3)$$
$$\sum_{i=1}^{2n}(2i^2-5i-3) - \sum_{i=1}^{n}(2i^2-5i-3)$$
$$\frac{2(2n)(2n+1)(4n+1)}{6} - \frac{5(2n)(2n+1)}{2} - 3(2n) - \frac{2n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2} + 3n$$
$$\frac{2(2n)(2n+1)(4n+1)}{6} - \frac{15(2n)(2n+1)}{6} - 3n - \frac{2n(n+1)(2n+1)}{6} + \frac{15n(n+1)}{6}$$
$$\frac{(2n)(2n+1)}{6}(2(4n+1)-15) - 3n - \frac{n(n+1)}{6}(2(2n+1)-15)$$
$$\frac{(2n)(2n+1)}{6}(8n-13) - 3n - \frac{n(n+1)}{6}(4n-13)$$
$$\frac{n}{6}[2(2n+1)(8n-13) - 18 - (n+1)(4n-13)]$$
$$\frac{n}{6}[32n^2 - 36n - 26 - 18 - (4n^2 - 9n - 13)]$$
$$\frac{n}{6}[32n^2 - 36n - 26 - 18 - 4n^2 + 9n + 13]$$
$$\boldsymbol{\frac{n}{6}[28n^2 - 27n - 31]}$$

8) Consider an arithmetic sequence with the first term 22 and the tenth term 67. What is the sum of the first 20 terms of the sequence?

Solution
Using the given information, letting the i^th term in the sequence be a_i and the common difference be d, we find:

$$a_{10} - a_1 = 67 - 22 = 9d$$
$$45 = 9d$$
$$d = 5$$

Thus, the common difference is 5 and $a_{20} = a_1 + 19d = 22 + 19(5) = 117$.

We find the sum of the first twenty terms by plugging in n = 20: $\frac{n(a_1+a_n)}{2} = \frac{20(22+117)}{2} = \mathbf{1390}$.

9) Calculate the matrix product $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^2$.

Solution

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{9} & \mathbf{4} & \mathbf{8} \\ \mathbf{12} & \mathbf{11} & \mathbf{10} \\ \mathbf{20} & \mathbf{12} & \mathbf{19} \end{bmatrix}$$

10) Calculate the following Boolean product $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$