COT 3100 Recitation #4: Number Theory Practice - Solutions
9/19-23/2016

Warm-Up Problems

1) For how many ordered pairs of positive integers (x, y) is 3x + 5y = 2016?

Solution

We can find the solution that maximizes y by plugging in 5y = 2015, followed by 5y = 2010. The second yields y = 402 with x = 2. In general, we see that if (x, y) is a solution, so is (x+5, y-3). So if we started listing some solutions, we'd get (2, 402), (7, 399), (12, 396), etc. The last of these solutions would be (667, 3). Notice that since the question asked for positive solutions only, (672, 0) doesn't count. The number of ordered pairs listed above is the same as the number of terms in an arithmetic sequence with the first term of 2, last term of 667 and a common difference of 5. This is simply equal to $\frac{667-2}{5} + 1 = \frac{665}{5} + 1 = 134$.

2) A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

Solution

Let r be the radius. If the diameter is increased by 25%, the radius is increased by 25% also. Let V be the current volume of the jar, r be the current radius and h be the current height. Thus, we have:

$$V = \pi r^2 h$$

Let the new radius be r' and the new height be h'. We know that $r' = 1.25r$, thus:

$$V = \pi r^2 h = \pi (1.25r)^2 h'$$
$$r^2 h = \frac{25}{16} r^2 h'$$
$$h' = \frac{16}{25} h$$

It follows that we must decrease the height by 36%, since 16 is 64% of 25.

3) The sum of 49 consecutive integers is 7^5. The median of these 49 consecutive integers is 7^k for some integer k. What is k?

Solution

When we have an odd number of consecutive integers, the median integer IS the average value. To see this, note that any list of odd length must have 2a+1 values for some non-negative integer a. In particular, these values are x - a, x - a + 1, ..., x - 1, x, x + 1, ..., x + a, for some integer x. Precisely a values are less than x and a values are more than x, making x the median. Further more, the sum of these values is (2a+1)x, thus the average of the values is simply x.

The average of the given integers is $\frac{7^5}{49} = \frac{7^5}{7^2} = 7^3$, and this is also the median. Thus, the desired value for k is 3.

4) The average value of all the pennies, nickels, dimes and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

Solution

Let the purse have p pennies, n nickels, d dimes and q quarters. The first piece of given information means:

$$p + 5n + 10d + 25q = 20(p + n + d + q)$$

The next piece of given information means:

$$p + 5n + 10d + 25q + 25 = 21(p + n + d + q + 1)$$

Subtract the top equation from the bottom to yield:

$$25 = p + n + d + q + 21$$
$$p + n + d + q = 4$$

Thus, she must have only 4 coins in her purse for a total of 80 cents. Adding a quarter does indeed increase the total to 105 cents for an average of 21 cents a coin. To solve the problem at hand, we must come up with a sequence of four coins that adds up to 80 cents. If she only had two quarters, the maximum sum would be 2 x 25 + 2 x 10 = 70 cents. This means she must have more than two quarters. Of course, similar logic yields that 4 quarters is impossible, so she must have three quarters and a nickel. It follows that she has no dimes.

5) A sequence of three real numbers forms an arithmetic progression with the first term 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression.

Solution

Let the terms be a, b and c, with a = 9, b = 9 + d and c = 9 + 2d, where d is the common differences of the arithmetic sequence. Let a' = a = 9, b' = b + 2 = 11 + d and c' = 20 + c = 29 + 2d. Setting equations to solve for the common ratio of the geometric sequence we see that:

$$\frac{d+11}{9} = \frac{2d+29}{d+11}$$

Cross multiply (valid as long as d isn't -11, which would be impossible if the second sequence is a valid geometric sequence with a non-zero common ratio) to yield:

$$18d + 261 = d^2 + 22d + 121$$
$$d^2 + 4d - 140 = 0$$
$$(d + 14)(d - 10) = 0$$

It follows that the common difference is either -14 or 10. Thus, the original arithmetic sequence is either 9, -5, -19 or 9, 19, 29. The corresponding geometric sequences would be 9, -3, 1 and 9, 21, 49. It follows that the smallest possible value for the third term of the geometric sequence is 1.

Number Theory Problems

6) Use the Euclidean Algorithm to find the greatest common divisor of 565 and 235.

Solution

$$565 = 2 \times 235 + 95$$
$$235 = 2 \times 95 + 45$$
$$95 = 2 \times 45 + 5$$
$$45 = 9 \times 5$$

It follows that gcd(565, 235) = 5.

7) For positive integers, *m*, find all integers *x* with $0 \leq x < 5$ such that

$$m^2 - m + 11 \equiv x(mod\ 5)$$

(Hint: If we plug in two different values of *m* that are equivalent mod 5, we will obtain the same result for *x*. Thus, it's enough to specifically plug in *m* = 0, 1, 2, 3 and 4 to obtain all possible results for *x*.)

Based on your list, is it possible for $m^2 - m + 11$ to be divisible by 5 if *m* is a positive integer?

Solution

Here is a table plugging in each value of *m* that may lead to a unique result:

m	m^2-m+11	$(m^2-m+11) \equiv x \bmod 5$
0	11	1
1	11	1
2	13	3
3	17	2
4	23	3

All possible integers *x* are 1, 2 and 3. It follows that this expression will never be divisible by 5 for any integer *m*.

8) Using the division algorithm repeatedly, convert 1431 in base 10 to base 7.

Solution

7 | 1431
7 | 204 R3
7 | 29 R1
7 | 4 R 1
7 | 0 R 4

Thus, we have $1431_{10} = 4113_7$.

9) Use the textbook's modular exponentiation algorithm to calculate the remainder when 2^{27} is divided by 11.

Solution

$$2^1 \equiv 2\, mod 11$$
$$2^2 \equiv 4 mod 11$$
$$2^4 \equiv 4^2 \equiv 5\, mod 11$$
$$2^8 \equiv 5^2 \equiv 3 mod 11$$
$$2^{16} \equiv 3^2 \equiv 9 mod 11$$

Thus, we have $2^{27} \equiv 2^{16}2^8 2^2 2^1 \equiv (9 \times 3 \times 4 \times 2) \equiv (9 \times 12 \times 2) \equiv 9 \times 1 \times 2 \equiv 7 (mod\ 11)$

10) Use the Sieve of Eratostenes to generate a list of all prime numbers less than 50.

Solution

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	~~25~~	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	~~35~~	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	~~49~~	~~50~~

The progression of the algorithm is shown above, each different number highlighted represents numbers circled to be prime. The numbers with ~~6~~ are the values crossed off during the iteration of the algorithm where 2 was circled, so all multiples of 2 (greater than 2). The numbers with 9 represent multiples of 3 that weren't previously crossed off. The numbers with ~~35~~ represent multiples of 5 not previously crossed off and both the single and double strikethroughs ~~49~~ represent multiples of 7 not previously crossed off.

Thus all of the primes from 2 to 50 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.