COT 3100: Exam #1 Review Solutions

1) Prove or disprove the following statement over the universe of all real numbers for x and y: $\exists x \forall y [y^2 - 6y + x \geq 0]$.

Solution
This statement is true for x = 9. We prove it as follows for this specific value:

$$y^2 - 6y + 9 = (y - 3)^2 \geq 0$$

Since y is real, y - 3 is also real, the square of any real number is non-negative, proving the assertion for x = 9. Thus, a real value of x exists to make the statement true.

More generally, the statement is true for all real x ≥ 9.

2) Prove or disprove the following assertion for finite sets A, B and C:

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Solution
This assertion is true. To prove that two sets are equal, we must show that each set is a subset of each other. Thus, we must show the two following things:

$$1. A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$$

$$2. (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$$

We use direct proof to show both.

To prove the first, we must show that for an arbitrary element $(x, y) \in A \times (B \cap C)$, that $(x, y) \in (A \times B) \cap (A \times C)$.

By definition of Cartesian Product, if $(x, y) \in A \times (B \cap C)$, then $x \in A \wedge y \in B \cap C$.
By definition of set intersection $y \in B \wedge y \in C$.
Since $x \in A \wedge y \in B$, by definition of Cartesian Product we conclude that $(x, y) \in (A \times B)$.
Since $x \in A \wedge y \in C$, by definition of Cartesian Product we conclude that $(x, y) \in (A \times C)$.
By definition of set intersection, we arrive at the conclusion: $(x, y) \in (A \times B) \cap (A \times C)$.

To prove the second, we must show that for an arbitrary element $(x, y) \in (A \times B) \cap (A \times C)$, that $(x, y) \in A \times (B \cap C)$.

By definition of set intersection, if $(x, y) \in (A \times B) \cap (A \times C)$, it follows that $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$.

By definition of Cartesian Product, applied to both of those statements above individually, we then find that $x \in A \wedge y \in B$, and $x \in A \wedge y \in C$.

By definition of set intersection, it follows that $y \in B \cap C$.

Since $x \in A \wedge y \in B \cap C$, by definition of Cartesian Product, it follows that $(x, y) \in A \times (B \cap C)$, as desired.

3) Use the laws of implication to prove the conclusion shown from the following premises:

$(p \vee q) \to (t)$
$s \to (r \vee p)$
$u \to (\bar{r} \vee q)$
$v \to (s \wedge u)$
v
----
t

Solution

Step	Reason
1. v	Premise
2. $v \to (s \wedge u)$	Premise
3. $s \wedge u$	Modus Ponens with steps 1 and 2
4. s	Conjunctive Simplification
5. $s \to (r \vee p)$	Premise
6. $r \vee p$	Modus Ponens with steps 4 and 5
7. u	Conjunctive Simplification
8. $u \to (\bar{r} \vee q)$	Premise
9. $\bar{r} \vee q$	Modus Ponens with steps 7 and 8
10. $p \vee q$	Resolution with steps 6 and 9
11. $(p \vee q) \to (t)$	Premise
12. t	Modus Ponens with steps 10 and 11

4) Prove or disprove the assertion below for finite sets A and B. (Note: $\wp(A)$ denotes the power set of A.)

$$\wp(A) \cup \wp(B) = \wp(A \cup B)$$

Solution

This claim is false. We can show this via counter-example. Consider the situation were $A = \{1\}$ and $B = \{2\}$. For this specific example, we have

$\wp(A) = \{\ \emptyset, \{1\}\}$, $\wp(B) = \{\ \emptyset, \{2\}\}$, so $\wp(A) \cup \wp(B) = \{\emptyset, \{1\}, \{2\}\}$.

But, $A \cup B = \{1,2\}$, thus, $= \wp(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

Thus, for this counter-example, we see that $\{1,2\} \in \wp(A \cup B)$, but $\{1,2\} \notin \wp(A) \cup \wp(B)$. Thus, the two sets are not equal as claimed.

Note: More generally, any counter-example where $(A - B \neq \emptyset) \wedge (B - A \neq \emptyset)$ will work, since if the following is true, there exists some x for which $x \in A \wedge x \notin B$ and there exists some y for which $y \notin A \wedge y \in B$. Then, we have that $\{x, y\} \in \wp(A \cup B)$ and $\{x, y\} \notin \wp(A) \cup \wp(B)$, proving that the two sets are not equal. Also, the set on the left IS a subset of the set on the right.

5) Find all ordered pairs of integers (x, y) such that 131x + 58y = 4.

Solution

Run the Extended Euclidean on 131 and 58:

131 = 2 x 58 + 15
58 = 3 x 15 + 13
15 = 1 x 13 + 2
13 = 6 x 2 + 1

13 - 6 x 2 = 1
13 - 6(15 - 13) = 1
13 - 6 x 15 + 6 x 13 = 1
7 x 13 - 6 x 15 = 1
7(58 - 3 x 15) - 6 x 15 = 1
7 x 58 - 21 x 15 - 6 x 15 = 1
7 x 58 - 27 x 15 = 1
7 x 58 - 27(131 - 2 x 58) = 1
7 x 58 - 27 x 131 + 54 x 58 = 1
61 x 58 - 27 x 131 = 1

Now, multiply this equation through by 4 to yield:

(61 x 4) x 58 + (-27 x 4) x 131 = 1

244 x 58 + (-108) x 131 = 1

Thus, one solution is x = -108, y = 244. Since GCD(131, 58) = 1, we can formulate all solutions with the following set of ordered pairs:

{(-108 + 58c, 244 - 131c) | ∈ *Z* }.

Alternatively, we can represent the same set as follows:

{(8+ 58c, -18 - 131c) | ∈ *Z* }.

6) Prove or disprove: Let a, b, c and d be arbitrary positive integers. If a | b and c | d, then ab | (c+d).

Solution

This is false, consider the following counter-example: a = 10, b = 10, c = 1, d = 2. For these numbers, it is the case that 10 | 10 and 1 | 2, but it's not the case that ab = 10(10) = 100 divides evenly into c + d = 3. Most examples that one tries would prove the statement false since the if portion of the statement makes no restriction on the relationship between the pair of numbers a and b and the pair of numbers c and d.

7) Let $n = 2^3 3^5 7^2$ and $m = 2^2 3^7 5^3$. Find gcd(n, m) and lcm(n, m) in prime factorized form.

Solution

gcd(n, m) = $\mathbf{2^2 3^5}$ and lcm(n, m) = $\mathbf{2^3 3^7 5^3 7^2}$**.**

8) Using Fermat's Theorem, determine the remainder when 17^{122} is divided by 61.

Solution

Since 61 is prime, Fermat's Theorem tells us that $17^{60} \equiv 1$ mod 61. Squaring this yields that

$17^{120} \equiv 1$ mod 61

Now, work out the given problem:

$17^{122} \equiv 17^{120} 17^2 \equiv 17^2 \equiv 289 \equiv$ **45 mod 61**