COT 3100 Recitation #2: Logic Practice
1/17-20/2017

Warm-Up Problems

1) Al gets the disease algebritis and must take one green pill and one pink pill each day for two weeks. A green pill costs $1 more than a pink pill and Al's pills cost a total of $546 for the two weeks. How must does one green pill cost?

Solution

Let the green pill cost g dollars. Then the pink pill costs g - 1 dollars. Thus we have:

$$14(g + g - 1) = 546$$
$$2g - 1 = 39$$
$$g = 20$$

One green pill costs **$20.**

2) The second and fourth terms of a geometric sequence are 2 and 6, respectively. What are the possible values of the first term in the sequence?

Solution

Let r be the common ratio of the sequence. The given information indicates that $r^2 = \frac{6}{2} = 3$. It follows that $r = \pm\sqrt{3}$. The two possible values of the first term are simply two (the second term) divided by the two possible common ratio:

$$\frac{2}{\pm\sqrt{3}} = \pm\boldsymbol{\frac{2\sqrt{3}}{3}}$$

3) Let s(x) denote the sum of the digits of the positive integer x. For example s(8) = 8 and s(123) = 1 + 2 + 3 = 6. For how many two-digit values of x is s(s(x)) = 3?

Solution

The question is simply asking for how many two digits numbers, x, is the sum of digits y, where the sum of digits of y is 3. The possible values for y are 3, 12, 21 and 30. Thus, we must list all 2 digits numbers that have a sum of digits of 3, 12, 21 or 30. The maximum sum of two digits is 18, so we must simply find all two digits numbers whose sum of digits is either 3 or 12. These are 12, 21, 30, 39, 48, 57, 66, 75, 84, and 93. Thus, there are **10** such values.

4) Cassandra sets her watch to the correct time at noon. At the actual time of 1:00 PM, she notices that her watch reads 12:57 and 36 seconds. Assuming that her watch loses time at a constant rate, what will be the actual time when her watch first reads 10:00 PM?

Solution
There are 3600 seconds in an hour. Cassandra's watch ticks $57 \times 60 + 36 = 3456$ seconds in this time. Thus, the actual time moves at a rate of $\frac{3600}{3456} = \frac{225}{216}$ compared to Cassandra's clock. Thus, the number of actual seconds that elapse by the time Cassandra's clock reads 10:00 PM (when it had ticked $10\ hr \times \frac{3600ticks}{hr} = 36000ticks$, can be calculated as follows:

$$\frac{225sec}{216ticks} \times 36000ticks = 225 \times \frac{1000}{6} sec \times \frac{1hr}{3600sec} = 10\frac{5}{12}hr$$

Since there are 60 minutes in an hour, this equates to **<u>10:25 PM</u>** for the actual time.

5) If $\log(xy^3) = 1$ and $\log(x^2y) = 1$ what is $\log(xy)$?

Solution
Let $a = \log(x)$ and $b = \log(y)$. We seek $\log(xy) = \log(x) + \log(y) = a + b$.

Write both equations out and simplify:

$$\log(xy^3) = 1$$
$$\log(x) + \log(y^3) = 1$$
$$\log(x) + 3\log(y) = 1$$
$$a + 3b = 1$$

$$\log(x^2y) = 1$$
$$\log(x^2) + \log(y) = 1$$
$$2\log(x) + \log(y) = 1$$
$$2a + b = 1$$

Solve the system for *b*:

$$2a + 6b = 2$$
$$2a + b = 1$$

$$5b = 1,\ b = \frac{1}{5}$$

Since $a + 3b = 1$, $a + 3(\frac{1}{5}) = 1$, so $a = \frac{2}{5}$. Finally, the sum of a and b, our desired quantity, is $\frac{1}{5} + \frac{2}{5} = \mathbf{\frac{3}{5}}$.

Logic Problems

6) Using the following premises:

$(p \wedge t) \rightarrow (r \vee s)$
$q \rightarrow (u \wedge t)$
$u \rightarrow p$
$\bar{s}$

Derive the conclusion $q \rightarrow r$.

Solution

1. $q \rightarrow (u \wedge t)$,	Given
2. $\bar{q} \vee (u \wedge t)$,	Definition of Implication
3. $(\bar{q} \vee u) \wedge (\bar{q} \vee t)$,	Distributive Law
4. $(\bar{q} \vee u)$,	Conjunctive Simplification with step 3
5. $q \rightarrow u$,	Definition of Implication with step 4
6. $u \rightarrow p$,	Given
7. $q \rightarrow p$,	Law of Syllogism with steps 5, 6
8. $(\bar{q} \vee t)$,	Conjunctive Simplification with step 3
9. $q \rightarrow t$,	Definition of Implication with step 8
10. $(\bar{q} \vee p)$,	Definition of Implication with step 7
11. $(\bar{q} \vee t) \wedge (\bar{q} \vee p)$,	Conjunction with steps 8 and 10
12. $\bar{q} \vee (p \wedge t)$,	Distributive Property with step 11
13. $q \rightarrow (p \wedge t)$,	Definition of Implication with step 12
14. $(p \wedge t) \rightarrow (r \vee s)$	Given
15. $q \rightarrow (r \vee s)$,	Law of Syllogism with steps 13, 14
16. $\bar{q} \vee (r \vee s)$	Definition of Implication with step 15
17. $(\bar{q} \vee r) \vee s$,	Associative Law step 16
18. $(q \rightarrow r) \vee s$	Definition of Implication with step 17
19. $\bar{s}$	Premise
20. $q \rightarrow r$	Rule of Disjuctive Syllogism with steps 18, 19

7) In class, Modus Ponens was proved using just the laws of logic. Prove Modus Tollens in the same manner.

Solution

We must show that $(\bar{q} \wedge (p \rightarrow q)) \rightarrow \bar{p}$ is a tautology via the Laws of Logic.

$\left(\bar{q} \wedge (p \rightarrow q)\right) \rightarrow \bar{p},$	Given
$\overline{(\bar{q} \wedge (p \rightarrow q))} \vee \bar{p},$	Definition of Implication
$(\bar{\bar{q}} \vee \overline{p \rightarrow q}) \vee \bar{p},$	De Morgan's Law
$(q \vee \overline{\bar{p} \vee q}) \vee \bar{p},$	Double Negation, Definition of Implication
$(q \vee (\bar{\bar{p}} \wedge \bar{q})) \vee \bar{p},$	De Morgan's Law
$(q \vee (p \wedge \bar{q})) \vee \bar{p},$	Double Negation
$((q \vee p) \wedge (q \vee \bar{q})) \vee \bar{p},$	Distributive Law
$((q \vee p) \wedge T) \vee \bar{p},$	Negation Laws
$(q \vee p) \vee \bar{p},$	Identity Law
$q \vee (p \vee \bar{p}),$	Associative Law
$q \vee T,$	Negation Law
$T,$	Domination Law

8) Simplify the following logical expression as much as possible using the laws of logic only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

$$p \vee [p \wedge [\neg(\neg r \vee \neg q) \vee (\neg r \wedge q)]]$$

Solution

$p \vee \left[p \wedge [\neg(\neg r \vee \neg q) \vee (\neg r \wedge q)]\right]$	$\leftrightarrow p \vee [p \wedge [(\neg\neg r \wedge \neg\neg q) \vee (\neg r \wedge q)]]$	DeMorgan's
	$\leftrightarrow p \vee [p \wedge [(r \wedge q) \vee (\neg r \wedge q)]]$	Double Negation
	$\leftrightarrow p \vee [p \wedge [(r \vee \neg r) \wedge q]]$	Distributive
	$\leftrightarrow p \vee [p \wedge [T \wedge q]]$	Inverse Law
	$\leftrightarrow p \vee [p \wedge q]$	Identity Law
	$\leftrightarrow p$	Absorption Law

9) Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used.

Solution

$(p \rightarrow r) \wedge (q \rightarrow r)$	
$= (\neg p \vee r) \wedge (\neg q \vee r)$	Definition of Implication
$= ((\neg p \vee r) \wedge \neg q) \vee ((\neg p \vee r) \wedge r)$	Distibutive Law
$= ((\neg p \vee r) \wedge \neg q) \vee r$	Absorption Law
$= (\neg p \wedge \neg q) \vee (r \wedge \neg q) \vee r$	Distibutive Law
$= \neg(p \vee q) \vee (r \wedge \neg q) \vee r$	DeMorgan's Law
$= \neg(p \vee q) \vee r$	Absorption Law
$= (p \vee q) \rightarrow r$	Definition of Implication

Here is an alternate and quicker solution:

$(p \rightarrow r) \wedge (q \rightarrow r)$	
$= (\neg p \vee r) \wedge (\neg q \vee r)$	Definition of Implication
$= (\neg p \wedge \neg q) \vee r$	Distributive Law
$= \neg(p \vee q) \vee r$	DeMorgan's Law
$= (p \vee q) \rightarrow r$	Definition of Implication

10) Use the Rules of Inference and the Law of Contraposition to validate the conclusion drawn below. (Each of the items above the dotted line is a premise, while the conclusion to draw is below the dotted line.) Show each step and state which rule is being used.

$q \rightarrow (u \wedge t)$
$u \rightarrow p$
q

$p \wedge t$

Solution

Step		Rule
1.	$q \rightarrow (u \wedge t)$	Premise
2.	q	Premise
3.	$(u \wedge t)$	Modus Ponens (1, 2)
4.	u	Simplification (3)
5.	$u \rightarrow p$	Premise
6.	p	Modus ponens (4, 5)
7.	t	Simplification (4)
$p \wedge t$	Conjunction (6, 7)	