Fall 2016 COT 3100 Section 1 Homework 7 Solutions

1) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4}.
In particular, let R_1 = {(1, 1), (1, 2), (2, 1), (3, 4), (4, 4)} and R_2 = {(1, 2), (2, 3), (3, 4)}.

Determine the following:

a) Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R_1 and R_2.

Solution

a)

R_1 is **not reflexive** since $(3, 3) \notin R_1$
R_1 is **not irreflexive** since $(1, 1) \in R_1$
R_1 is **not symmetric** since $(3, 4) \in R_1$, but $(4, 3) \notin R_1$
R_1 is **not anti-symmetric** since $(1, 2) \in R_1$ and $(2, 1) \in R_1$
R_1 is **not transitive**, since $(2, 1) \in R_1$, $(1, 2) \in R_1$, but $(2, 2) \notin R_1$

b)

R_2 is **not reflexive** since $(1, 1) \notin R_2$
R_2 is **irreflexive** since $\nexists i \,|(i \in A \land (i, i) \in R_2)$
R_2 is **not symmetric** since $(3, 4) \in R_1$, but $(4, 3) \notin R_2$
R_2 is **anti-symmetric** since $\nexists i\,, j|(i, j \in A \land (i, j) \in R_2 \land (j, i) \in R_2)$
R_2 is **not transitive**, since $(1,2) \in R_2$, $(2,3) \in R_2$, but $(1,3) \notin R_2$

c) $(1, 2) \in R_2$, and $(2, 1) \in R_1$, thus $(1, 1) \in R_1 \circ R_2$
$(2, 3) \in R_2$, and $(3, 4) \in R_1$, thus $(2, 4) \in R_1 \circ R_2$
$(3, 4) \in R_2$, and $(4, 4) \in R_1$, thus $(3, 4) \in R_1 \circ R_2$
$R_1 \circ R_2$: $\{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{4})\}$

d) $(1, 1) \in R_1$, and $(1, 2) \in R_2$, thus $(1, 2) \in R_2 \circ R_1$
$(1, 2) \in R_1$, and $(2, 3) \in R_2$, thus $(1, 3) \in R_2 \circ R_1$
$(2, 1) \in R_1$, and $(1, 2) \in R_2$, thus $(2, 2) \in R_2 \circ R_1$
$(3, 4) \in R_1$, but $(4, i) \notin R_2$, thus $(3, i) \notin R_2 \circ R_1$
$(4, 4) \in R_1$, but $(4, i) \notin R_2$, thus $(4, i) \notin R_2 \circ R_1$
$R_2 \circ R_1$: $\{(\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), (\mathbf{2}, \mathbf{2})\}$

e) $R_1 \cup R_2$: **{(1, 1), (1, 2), (2, 1), (2, 3), (3, 4), (4, 4)}**

f) $R_1 \cap R_2$: **{(1, 2), (3, 4)}**

g) $r(R_1)$: **{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 4)}**
$s(R_1)$: **{(1, 1), (1, 2), (2, 1), (3, 4), (4, 3), (4, 4)}**
$t(R_1)$: **{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 4)}**

$r(R_2)$: **{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3) (3, 4), (4, 4)}**
$s(R_2)$: **{(1, 2), (2, 1), (2, 3), (3, 2), (3, 4), (4, 3)}**
$t(R_2)$: **{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}**

2) Let R be a relation on the set Z^+ defined as follows:

R = {(a, b) | $\exists c \in Z^+ \text{ such that } c^2 = a^2 + b^2$}

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

Solution:

R is **not reflexive** since (1, 1)∉R. $1^2 + 1^2 = 2$. Since $\sqrt{2}$ is irrational, $\nexists c \in Z^+ | \, c^2 = 2$.
R is **irreflexive**. Assume to the contrary: $\exists a | (a, a) \in R$
This means $\exists c \in Z^+ | \, c^2 = a^2 + a^2$
Then $2a^2 = c^2$.
Thus $a^2 = \frac{c^2}{2}$.
Then $\sqrt{a^2} = \frac{\sqrt{c^2}}{\sqrt{2}} \to \sqrt{2} = \frac{c}{a}$.
But $a, c \in Z^+$. This contradicts the fact that $\sqrt{2}$ is irrational. Thus $\nexists a | (a, a) \in R$.

R is **symmetric**, since $(a, b) \in R \to \exists c \in Z^+ | c^2 = a^2 + b^2$, using commutative property of addition, $b^2 + a^2 = c^2$. Thus, $(b, a) \in R$

R is **not antisymmetric**, since $(3,4) \in R \text{ and } (4,3) \in R$. ($3^2 + 4^2 = 5^2$, and vice versa)

R is **not transitive**, as $(20,15) \in R \wedge (15, 36) \in R \wedge (20,36) \notin R$.

$$20^2 + 15^2 = 25^2$$
$$15^2 + 36^2 = 39^2$$
$$20^2 + 36^2 = 1696, \sqrt{1696} \notin Z^+$$

3) Let b(n) equal the number of bits in the binary representation of the positive integer n. Prove that the relation, R, defined below is an equivalence relation. Into how many equivalence classes does R partition the set of integers? Explicitly list all of the members of the following equivalence classes: [2], [12] and [27]. How many elements are in the equivalence class [2^k] for any positive integer k, in terms of k?

R = {(x, y) | $b(x) = b(y)$}

Solution
R is reflexive, as $b(i) = b(i)$ for all $i \in Z$.
R is symmetric, as $b(i) = b(j) \rightarrow b(j) = b(i)$
R is transitive, as if $b(i) = b(j) \wedge b(j) = b(k), then\ b(i) = b(k)$
Thus R is an equivalence relation.

R partitions the set of integers into infinite classes.

[2] = {2, 3} (2= 10 binary, b(2) = 2, b(3) = 2)
[12]= {8,9,10,11,12,13,14,15} (12 = 1100 in binary, b(12)=4)
[27]={16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31}(27=11011 in binary, b(27)=5)
$|2^k| =$, since
2^kcan be represented as 1 followed by k zeroes. Thus b(2^k) $= k + 1$
The number of positive integers that can be represented with $k + 1$ digits is 2^k(the first digit is 1to preserve length, 2^k choices remain).
Thus $|[2^k]| = 2^k$.

4) Let R be a relation on the set Z^+ defined as follows:

R = {(a, b) | |a - b| ≤ 5}

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

Solution
R is **reflexive**, as $a - a = 0, |0| \leq 5$, thus $(a, a) \in R$
R is **not irreflexive**, since $(1,1) \in R$
R is **symmetric**, since $(a, b) \in R \rightarrow a - b = c \wedge\ |c| \leq 5$. Thus,
$b - a = -c \wedge\ |-c| = |c| \wedge |-c| \leq 5$. Thus, $(a, b) \in R \rightarrow (b, a) \in R$.
R is **not antisymmetric,** since $(2,1) \in R \wedge (1,2) \in R$
R is **not transitive,** since $(1,6) \in R \wedge (6,7) \in R$, but $(1,7) \notin R$

$$1 - 6 = -5, |-5| = 5 \leq 5$$
$$6 - 7 = -1, |-1| = 1 \leq 5$$
$$1 - 7 = -6, |-6| = 6 \nleq 5$$

5) Let R be a relation on the set $Z^+ \times Z^+$ defined as follows:

R = { ((a,b) , (c,d)) | a + b ≥ c + d }.

Is R a partial-ordering relation? Why or why not?

Define S = { ((a,b) , (c,d)) | a + b = c + d } also on the set $Z^+ \times Z^+$. Is S an equivalence relation? Prove your answer.

Solution

For R to be a partial ordering relation, it has to be reflexive, anti-symmetric, and transitive.
Note that $((1,3),(2,2)) \in R$, since $1 + 3 \geq 2 + 2$. Note that $\big((2,2),(1,3)\big) \in R$ as well. Since $(1,3) \neq (2,2)$, R is not anti-symmetric. Thus, R is not a partial ordering relation.

For S to be an equivalence relation, it has to be reflexive, symmetric, and transitive.

S is reflexive, as $a + b = a + b$ for all a, b. Thus $\big((a,b),(a,b)\big) \in S$.
S is symmetric, as $a + b = c + d \rightarrow c + d = a + b$. Thus
$\big((a,b),(c,d)\big) \in S \rightarrow \big((c,d),(a,b)\big) \in S$
S is transitive. If $\big((a,b),(c,d)\big) \in S$, then $a + b = m = c + d$. If $\big((c,d),(e,f)\big) \in S$ as well, then $c + d = m = e + f$. Since $a + b = m$ and $e + f = m$, then $a + b = e + f$, meaning $\big((a,b),(e,f)\big) \in S$.

Thus, S is an equivalence relation.

6) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6} contain the ordered pairs (2, 2), (3, 4) and (5, 6)?

Solution

Let us consider possible relation R.
Note there are $\binom{6}{2} = \frac{6!}{2!4!} = 15$ pairs of distinct elements in A. Note that for each pair, $i, j \in A | i < j$ there are three possibilities: $(i,j) \in R$; $(j,i) \in R$; or neither is in R. Otherwise, R would not be antisymmetric. Of the 15 pairs, two are already accounted for: (3,4) and (5,6). The remaining 13 pairs have 3 choices. Thus there are 3^{13} choices regarding pairs of distinct elements.

Thus, there remains the possibilities (i,i) elements of R. Since (2,2) must be a part of R, there remain 5 (i,i) elements that are either in R or not in R. Thus there are 2^5 choices to be made regarding reflexive elements.

The choices are independent of one another, so use the multiplication rule to get $\underline{\mathbf{3^{13} * 2^5}}$ relations.

7) Let f(x) = $x^2 + 4x - 32$ with a domain of all real $x \in [-\infty, -2]$. Prove that f is injective. What is the range of f? (You may either use Calculus or complete the square to prove your answers.)

Solution

Note the roots of f(x) are $x = \frac{-4 \pm \sqrt{4^2 - (4*1*-32)}}{2} = -8{,}4$

Since this is a (upwards facing) parabola, the minimum point is between the two roots, at $x = \frac{-8+4}{2} = -2$.

Another way to find the minimum is to take the derivative of f: $f'(x) = 2x + 4$. Local minimums are found when f'(x) = 0: $2x + 4 = 0; x = -2$.

Since the domain of f is $[-\infty, 2]$, and no values are repeated on the same side of a parabola, the function is surjective.

Another way to think of this is to consider the derivative of f again. From $[-\infty, 2], f'(x)$ is negative. Thus it never "turns around" to repeat a value in the range of the function. Thus f is surjective.

The range of f starts at infinity when $x = -\infty$. The minimum, as we found, was at $x = -2$. Thus, finding the lower bound of the range can be found by plugging -2 into x:

$$(-2)^2 + 4(-2) - 32 = -36$$

Thus the range is $\underline{[\boldsymbol{-36}, \infty]}$

Without calculus we have:

f(x) = $x^2 + 4x - 32 = x^2 + 4x + 4 - 32 - 4 = (x + 2)^2 - 36$.

We must prove for two real values x and y, if x ≤ -2 and y ≤ -2, and f(x) = f(y), then x = y.

f(x) = f(y)
$(x + 2)^2 - 36 = (y + 2)^2 - 36$
$(x + 2)^2 = (y + 2)^2$

Now, take the square root of both sides to get:

$$x + 2 = \pm(y + 2)$$

But, since we KNOW that x ≤ -2, we know that the left-hand side is non-positive. Since y ≤ -2, it follows that the quantity in the parentheses is also non-positive. If we negate a non-positive quantity that isn't 0, we get a positive one, which is not allowed. This means that of the two choices (plus or minus), the minus is not possible. Thus, we now have:

x + 2 = y + 2
x = y, as desired.

8) Let A be a set of 10 elements and B be a set of 15 elements. How many functions can be defined with the domain of A and the co-domain of B?

Solution

Each element in A must correspond to an element of B. Thus each element of A has 15 choices of elements in B. Each choice is independent to another, so by using the multiplication rule: $\mathbf{15^{10}}$

9) Let f(x) = $3x^3$ + 2x - 7 and g(x) = 4x - 5. Determine f(g(x)) and g(f(x)).

Solution

$$f(g(x)) = f(4x-5) = 3(4x-5)^3 + 2(4x-5) - 7$$
$$= 3(64x^3 - 240x^2 + 300x - 125) + 8x - 10 - 7$$
$$= \mathbf{192x^3 - 720x^2 + 908x - 392}$$

$$g(f(x)) = g(3x^3 + 2x - 7) = 4(3x^3 + 2x - 7) + 7 = 12x^3 + 8x - 28 + 7$$
$$= \mathbf{12x^3 + 8x - 21}$$

10) Find $f^{-1}(x)$ for the function given in question #7.

Solution

Since we discovered f was an injunctive function, it has an inverse.
Since y = f(x) = x^2 + 4x – 32, the inverse can be found by swapping the y and xs, and solving for y.

$$x = y^2 + 4y - 32$$
$$x + 36 = y^2 + 4y + 4$$
$$x + 36 = (y+2)^2$$
$$\pm\sqrt{x+36} = (y+2)$$

Thus $y = -2 + \sqrt{x+36}$ or $y = -2 - \sqrt{x+36}$.

Since the range of f^{-1} should be the domain of f, which was stated to be $[-\infty, -2]$, we must make sure the values of y don't exceed -2. This can be accomplished by picking the second of the two possible equations. Thus the correct equation for $f^{-1}(x) = \mathbf{-2 - \sqrt{x+36}.}$

11) Let f(x) = 3x + 5. Let $f^n(x)$ to be the function f composed with itself n times. (For example, $f^3(x) = f(f(f(x)))$.) Using induction on n, prove that $f^n(x) = 3^n x + \frac{5}{2}(3^n - 1)$, for all positive integers n.

Solution

Base case: n = 1

$$f^1(x) = f(x) = 3x + 5$$
$$3x + \frac{5}{2} * 2$$
$$3^1 x + \frac{5}{2} * (3^1 - 1)$$

Thus proving the base case

Inductive Hypothesis: assume that for a positive integer > 1, k, that $f^k(x) = 3^k x + \frac{5}{2}(3^k - 1)$

Inductive step: Attempt to prove $f^{k+1}(x) = 3^{k+1} x + \frac{5}{2}(3^{k+1} - 1)$

$f^{k+1}(x) = f\left(f^k(x)\right)$	By definition of $f^n(x)$
$= f(3^k x + \frac{5}{2}(3^k - 1))$	By Inductive hypothesis
$= 3\left(3^k x + \frac{5}{2}(3^k - 1)\right) + 5$	By definition of f(x)
$= 3^{k+1} x + \frac{5}{2} * 3(3^k - 1) + 5$	Distribution
$= 3^{k+1} x + \frac{5}{2} * (3^{k+1} - 3) + 5$	Distribution
$= 3^{k+1} x + \frac{5}{2} * \left((3^{k+1} - 1) - 2\right) + 5$	Expansion
$= 3^{k+1} x + \frac{5}{2} * (3^{k+1} - 1) - \frac{5}{2} * (2) + 5$	Distribution
$= 3^{k+1} x + \frac{5}{2} * (3^{k+1} - 1) - 5 + 5$	Multiplication
$= 3^{k+1} x + \frac{5}{2} * (3^{k+1} - 1)$	Addition

Thus proving the Inductive Step. Thus by proving both the base case and the inductive step, $f^{k+1}(x) = 3^{k+1} x + \frac{5}{2}(3^{k+1} - 1)$ has been proven.