Fall 2016 COT 3100 Section 1 Homework 6 Solutions

1) In alpha testing a new software package, a software engineer finds that the number of defects per 100 lines of code is a random variable X with probability distribution:

x	1	2	3	4
Pr(X = x)	0.5	0.3	0.15	0.05

Find E(X) and Var(X).

<u>Solution</u>
The expected value and variance can be calculated form following equations:

$$E(X) = \sum_{x \in X} x * P(x)$$

$$Var(X) = \sum_{x \in X} (x - E(x))^2 * P(x)$$

So, the expected value will be:

E(X)=1*0.5+2*0.3+3*0.15+4*0.05=**<u>1.75</u>**

and then, the variance will be:

Var(X)= $(1-1.75)^2 * 0.5 + (2-1.75)^2 * 0.3 + (3-1.75)^2 * 0.15 +(4-1.75)^2 * 0.05$ = **<u>0.7875</u>**

2) A lottery allows a player to choose 6 values out of 60. The goal of the lottery is to break even exactly, making its cash prizes equal to the amount of money spent by the contestants. Each ticket costs $1 to buy and players receive winnings if they match 3, 4, 5 or all 6 numbers. If the payout for matching 3 numbers is $10, matching 4 numbers is $100, and matching 5 numbers is $10,000, what does the payout for matching all 6 numbers need to be?

<u>Solution</u>
First, we should calculate the probability of each matching (3,4,5 and 6). Then since the expected value should be equal to one for all the matching, we can calculate the price for matching 6 numbers.
Let's consider the n as the matching number (n= 3,4,5,6). So, the probability of each of them can be calculated as following:

$$P(3) = \frac{\binom{6}{3} * \binom{54}{3}}{\binom{60}{6}} = \frac{496080}{50063860}$$

$$P(4) = \frac{\binom{6}{4} * \binom{54}{2}}{\binom{60}{6}} = \frac{21465}{50063860}$$
$$P(5) = \frac{\binom{6}{5} * \binom{54}{1}}{\binom{60}{6}} = \frac{324}{50063860}$$
$$P(6) = \frac{1}{\binom{60}{6}} = \frac{1}{50063860}$$

E(X)=P(3)*reward(3)+ P(4)*reward(4)+ P(5)*reward(5)+ P(6)*reward(6)

E(X)= (1/50063860)*(496080*10+21465*100+324*10000+1*reward(6))

E(X)=1

So,

1= (1/50063860)*(496080*10+21465*100+324*10000+1*reward (6))
reward (6) = **$39716560**

3) The probability that it rains during a summer's day in a certain town is 0.3. In this town, the probability that the daily maximum temperature exceeds 25 degrees Celsius is 0.4 when it rains and 0.7 when it does not rain. Given that the maximum daily temperature exceeded 25 degrees Celsius on a particular summer's day, find the probability that it rained on that day.

Solution
Let's consider these variables for the events as following:

X: it rains
Y: temperature exceeds 25 degree

Then, we want to calculate the probability of P(X|Y) which according to Bayes rule we can write it as:

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

so, we can say:

P(X)=0.3
P(Y|X)=0.4
P(Y|~X)=0.7
but still we need P(Y). I order to calculate the P(Y) we can say that:

P(Y) = P(X) P(Y|X) + P(~X) P(Y|~X) = 0.3 * 0.4 + (1-0.3) * 0.7= 0.12 + 0.49 = 0.61

Now, we can use Bayes rule to calculate the P(X|Y):

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{0.4 * 0.3}{0.61} = \frac{\mathbf{12}}{\mathbf{61}}$$

4) Derek has a 32% chance of making exactly 1 free throw out of 2 free throws? What is his chance of making a single free throw? (Note: There are two possible answers.)

Solution

Consider P1 as a chance of making 1 throw out of 2 throw. And also, assume the probability of first throw is X. So, we can say that:

P1 = (2C1) * X* (1-X)

so:

0.32 = 2X(1-X) => 2X-2X^2 – 0.32 = 0
2X^2 - 2X + 0.32 = 0
X^2 - X + 0.16 = 0
(X - 0.8)(X - 0.2) = 0
=> **(X=0.8) or (X=0.2)**

5) In Arup's Game of Dice 2, you roll a fair pair of six-sided dice and record the total. If this total is 2, 4 or 12, you win. If it's a 5 or 11, you lose. In all other cases, you roll the pair of dice again. If the sum of this second roll exceeds the sum of your first roll, you win! Otherwise you lose. (For example, if you roll a 5 followed by a 6, you win, but if you roll a 10 followed by another 10, you lose.) What is the probability of winning Arup's Game of Dice?

Solution

The possibilities of winning in first round are : (1,1) , (1,3) , (3,1) , (2,2) , (6,6).

So, the probability of winning (P1) in first round is:

P1= (probability of each state) * (number of possibilities) = (1/36) * 5=5/36 (I)

Now, we should calculate the probability of winning in round 2. Winning in round 2 depends on what was rolled first. First, we calculate the number of states in round 1 that can lead to win in round 2. Here is all summation in round 1 that can permit the player to go to round 2: {3,6,7,8,9,10}

In addition, for winning, we need that total of second round exceeds that first round. In order to calculate this we can use following formulation:

$$\sum_{i\in\{3,6,7,8,9,10\}} (P(i) * (1 - \sum_{j=2}^{i} P(j)))$$

Now, we need to calculate the probability of each state:

P(2) = 1/36 , P(3) = 2/36, P(4) =3/36 , P(5) = 4/36, P(6) =5/36 , P(7) = 6/36, P(8) = 5/36, P(9) = 4/36 , P(10) = 3/36

Then:

P2 = $\sum_{i\in\{3,6,7,8,9,10\}}(P(i) * (1 - \sum_{j=2}^{i} P(j)))$

$= \frac{2}{36} * \frac{33}{36} + \frac{5}{36} * \frac{21}{36} + \frac{6}{36} * \frac{15}{36} + \frac{5}{36} * \frac{10}{36} + \frac{4}{36} * \frac{6}{36} + \frac{3}{36} * \frac{3}{36} = \frac{344}{36*36}$ (II)

In order to calculate the final probability of winning we should add (I) and (II):

P(win) = P1 +P2 = $\frac{5}{36} + \frac{344}{36*36} = \frac{524}{1296} = \frac{131}{324} \sim \mathbf{.404}$

6) Kellogs makes 27 types of cereal. You are sent a random box out of these 27 six months in a row. What is the probability that you receive at least one repeated box?

<u>Solution</u>

This is the birthday paradox. We calculate the opposite probability, that all 6 boxes of cereal are unique:

$$\frac{27}{27} \times \frac{26}{27} \times \frac{25}{27} \times \frac{24}{27} \times \frac{23}{27} \times \frac{22}{27} \sim .550$$

It follows that the probability some two boxes are the same is one minus this value, or roughly **<u>.450.</u>**

7) Suppose that one person in 10,000 people has a rare genetic disease. There is an excellent test for the disease; 99.6% of the people with the disease test positive and only 0.09% of the people who don't have it test positive. What is the probability that someone who tests positive has the disease? What is the probability that someone who tests negative does not have the disease?

<u>Solution</u>

First, let X be the event someone has a disease and Y be the event that someone tests positive for the disease. Then the given information can be written as follows:

P(X) = 1/10000
P(Y|X) = 0.996
P(Y|~X) = 0.0009

So, in order to calculate P(X|Y), we can use the Bayes rule as follows:

P(X|Y) = $P(X|Y) = \frac{P(X|Y)*P(X)}{P(Y)}$

For calculating P(Y), we can use following formulation:

P(Y) = P(X) P(Y|X) + P(~X) P(Y|~X) = 0.0001 * 0.996 + (1-0.0001) * 0.0009 = 0.00099951

so:

$$P(X|Y) = \frac{0.996 * 0.0001}{0.00099951} = \mathbf{0.0996}$$

Now, we want to calculate the probability of someone who test negative does not have disease. In order to calculate this we can conditional probability as following:

$$P(\bar{X}|\bar{Y}) = \frac{P(\bar{X} \cap \bar{Y})}{P(\bar{Y})}$$

then:

$$P(\bar{Y}) = P(\bar{Y} \cap X) + P(\bar{Y} \cap \bar{X}) = .0001 \times .004 + .9999 \times .9991 = .99900049$$

so:

$$P(\bar{X}|\bar{Y}) = \frac{0.9999 * 0.9991}{0.99900049} \sim \mathbf{.9999996}$$

8) Suppose we flip a fair coin until we get the same result two times in a row. What is the total number of times we are expected to flip the coin?

<u>Solution</u>

For any fixed number of coin flips, k, there are precisely two sequences of k coin flips that end in the same result the last two flips for the first time. For example, for k = 5, these two sequences are HTHTT and THTHH. In general, the sequence can end in either TT or HH. All preceding tosses must alternate in sequence. Thus, for any value k, k ≥ 2, the probability that we will flip exactly that number of times is $\frac{2}{2^k} = \frac{1}{2^{k-1}}$. Using the definition of expectation, the expected number of coin flips will be $\sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = \left(\sum_{k=1}^{\infty} \frac{k}{2^{k-1}}\right) - 1 = \frac{1}{\left(1-\frac{1}{2}\right)^2} - 1 = \frac{1}{\frac{1^2}{2}} - 1 = 2^2 - 1 = \mathbf{3}$

An alternate solution is to let E be the expected number of coin flips to play the game in the situation where there was a previous flip and the game wasn't over. In this case, we can set up an equation for E:

$$E = \frac{1}{2} \times 1 + \frac{1}{2} \times (1 + E)$$

The reasoning behind this equation is as follows: half of the time on a single flip, you'll match the previous flip, finishing the game. The other half of the time, you have used one flip and then are back to square one, playing the game from the beginning so your expected number of flips in this case is E + 1. Solving for E we get:

$$E = \frac{1}{2} + \frac{1}{2} + \frac{E}{2}$$
$$\frac{E}{2} = 1$$
$$E = 2$$

In the original version of the game, there is no previous flip. You MUST make that first flip. What follows is the adjusted game whose expected number of flips we just calculated. Thus, the expected number of flips for the original game is 2 + 1 = **3**.

9) Suppose E and F are events in a sample space and p(E) = 2/3, p(F) = 3/4, and p(F | E) =5/8. Find p(E | F).

Solution

According to Bayes rule we can say:

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

So:

$$P(E|F) = \frac{\left(\frac{5}{8}\right) * \left(\frac{2}{3}\right)}{\frac{3}{4}} = \mathbf{\frac{5}{9}}$$

10) What is the probability that each player has a hand containing an ace when each of four players receives 13 cards from the standard deck of 52? (Note: No two players will have the same card.)

Solution

We can view this problem as the probability of success of 4 items in a row: player 1 pulling 13 cards out of 52 and receiving one ace, then player two pulling 13 cards out of 39 and receiving one ace, followed by player 3 pulling 13 cards from 26 and receiving one ace, followed by player 4 pulling 13 cards of 13 remaining and receiving one ace. If there are k cards left, of which m are aces, the player in question can choose $\binom{k}{13}$ possible hands, of which $\binom{m}{1}\binom{k-m}{12}$ are the number of hands with exactly one ace. Thus, our desired probability is:

$$P = \left((4C1) * \frac{(48\ C\ 12)}{(52\ C\ 13)}\right) * \left((3C1) * \frac{(36\ C\ 12)}{(39\ C\ 13)}\right) * \left((2C1) * \frac{(24\ C\ 12)}{(26\ C\ 13)}\right) * \left((1C1) * \frac{(12\ C\ 12)}{(13\ C\ 13)}\right) \sim \mathbf{0.105}$$