

**Fall 2016 COT 3100 Section 1 Homework 5**

**Assigned: 10/28/2016**

**Due: 11/4/2016**

**Note: Please justify your answers and why you use each formula.**

1) Consider an ant that is walking on a Cartesian grid, starting at (0,0) and ending at (20, 12). The ant always chooses to walk exactly one unit either up or to the right (towards his destination) whenever he arrives at a Lattice point. (A Lattice point is a point with integer coordinates.) Thus, from (0,0) he either walks to (1, 0) or (0, 1). If the ant is not allowed to go to the points (10, 5) and (12, 8), how many different paths can he take on his walk?

**Solution**

Let the character 'R' represent moving to the right, and 'U' as moving up on the grid. Since, in order to reach the goal, we have to move right 20 times, and up 12 times, the problem reduces to finding the number of rearrangements of a string containing 20 'R's and 12 'U's. The number of such arrangements is  $\frac{32!}{20!12!}$ . This can also be analyzed as

**Going from (0,0) to (20, 12):**

The number of steps the ant must take moving up is 12

The number of steps the ant must take moving right is 20

The total number of ways the ant can get to (20,12) is  $\binom{32}{12}$

Next, let us count how many paths there are to get to the forbidden points and then subtract (or remove) those paths from the total  $\binom{32}{12}$ . There is one catch, though, and we will see what it is when we get to it.

**Going from (0,0) to (20, 12) via (10, 5):**

From (0,0) to (10, 5):

The number of steps the ant must take moving up is 5

The number of steps the ant must take moving right is 10

The total number of ways the ant can get to (10,5) is  $\binom{15}{5}$

Next, to get from (10, 5) to (20, 12),

The number of steps the ant must take moving up is 7

The number of steps the ant must take moving right is 10

The total number of ways the ant can get to (20,12) is  $\binom{17}{7}$

Therefore the total number of ways is  $\binom{15}{5} \binom{17}{7}$

Going from (0,0) to (20, 12) via (12, 8):

From (0,0) to (12, 8):

The number of steps the ant must take moving up is 8  
The number of steps the ant must take moving right is 12

The total number of ways the ant can get to (12,8) is  $\binom{20}{8}$

Next, to get from (12, 8) to (20, 12),

The number of steps the ant must take moving up is 8  
The number of steps the ant must take moving right is 4

The total number of ways the ant can get to (20,12) is  $\binom{12}{8}$

Therefore the total number of ways is  $\binom{20}{8} \binom{12}{8}$

**Now, consider removing these paths to the forbidden points:**

$$\binom{32}{12} - \binom{15}{5} \binom{17}{7} - \binom{20}{8} \binom{12}{8}$$

On removing these paths, we ended up removing paths that go through both (10, 5) and (12, 8) to the destination (20, 12) twice. We have to add back using the inclusion-exclusion principle.

To get from (10,5) to (12, 8)

The number of steps the ant must take moving up is 2  
The number of steps the ant must take moving right is 3

The total number of ways is  $\binom{5}{2}$

Here is how to put these together. First, we go from (0,0) to (10,5), then from (10,5) to (12,8), and then finally from (12,8) to (20, 12)

$$\binom{15}{5} \binom{5}{2} \binom{12}{8}$$

Finally, the total number of paths are

$$\binom{32}{12} - \binom{15}{5} \binom{17}{7} - \binom{20}{8} \binom{12}{8} + \binom{15}{5} \binom{5}{2} \binom{12}{8}$$

2) This question considered permutations of “HILLARYCLINTON”.

a) How many permutations are there total?

**Solution**

The 14 letters in the string can be arranged in  $14!$  ways, without counting the duplicates.

There are 2 duplicates of ‘I’, 3 duplicates of ‘L’, and 2 duplicates of ‘N’, so we divide the number of arrangements ( $14!$ ) by these to remove the over counting they caused.

$$\text{Total permutations} = \frac{14!}{2!3!2!}$$

b) How many permutations start and end with vowels?

**Solution**

The vowels in the string are: I, I, A, O

Breaking this up into different start and end vowel combinations, we have the following cases:

1. When both the start and the end are I’s, there is only one possible unique arrangement of the start and the end, and when arranging the rest of the letters, there are only duplicates of ‘L’ and ‘N’ to remove, giving

$$\frac{12!}{3!2!}$$

2. When the start and end vowel pair is ‘A’ and ‘O’, there are two possible arrangements (A...O and O...A). When arranging the remaining letters, we have to remove duplicates of ‘I’, ‘L’, and ‘N’, giving

$$2 * \frac{12!}{2!3!2!}$$

3. When the start and end vowel pair is ‘A’ and ‘I’, there are two possible arrangements (A...I, I...A). The remaining one ‘I’ goes into the permutation and does not get removed. The duplicates of ‘L’ and ‘N’ have to be removed. These considerations give us

$$2 * \frac{12!}{3!2!}$$

4. When the start and end vowel pair is ‘O’ and ‘I’, there are two possible arrangements (O...I, I...O). The remaining one ‘I’ goes into the permutation and does not get removed. The duplicates of ‘L’ and ‘N’ have to be removed. These considerations give us

$$2 * \frac{12!}{3!2!}$$

The final answer is the sum of these counts since these cases are disjoint:

$$\frac{12!}{3!2!} + 2 * \frac{12!}{2!3!2!} + 2 * \frac{12!}{3!2!} + 2 * \frac{12!}{3!2!}$$

A more succinct way to solve the problem is to see that we must put vowels in the first and last location. We only have the freedom to choose the locations of 2 vowels and we can do this in  $\binom{12}{2}$  ways. Once their positions are fixed, we can permute the vowels in  $\frac{4!}{2!}$  ways, since we have 4 vowels, one appearing twice. Finally, the positions of the consonants are fixed, but we can permute these in  $\frac{10!}{2!3!}$  ways. We multiply each of these to get a final answer of  $\binom{12}{2} \frac{4!10!}{2!2!3!}$  ways.

c) How many permutations do NOT have consecutive vowels in them?

**Solution**

This problem is equivalent to placing the vowels on the lines between the 10 consonants in the string

$$|C_1/C_2| \dots |C_{10}|$$

where  $C_i$  is the  $i$ -th consonant.

Since there are  $(10+1) = 11$  places to put a vowel, 4 vowels in the string, and we want to ensure that no spot is taken more than once, the total number of such choices is  $\binom{11}{4}$

Now we need to account for the rearrangement of vowels and consonants.

The total number of unique rearrangements of vowels is  $\frac{4!}{2!}$ , the number of total permutations of 4 vowels divided by the number of duplicate 2 'I' occurrences.

The total number of unique rearrangements of consonants is  $\frac{10!}{3!2!}$ , the number of total permutations of 10 consonants divided by the number of duplicate occurrences of 'L' and 'N'.

Thus, our final answer is the product of these values:

$$\binom{11}{4} * \frac{4!}{2!} * \frac{10!}{3!2!}$$

d) How many permutations are the letters in alphabetical order?

**Solution**

The alphabetic ordering of the letters is as follows: A, C, H, I, L, N, O, R, T, Y.

The number of permutations where the letters are in alphabetical order will be all of the permutations of ACHIILLNNORTY. There is just one such permutation since the duplicate letters I, L, and N are indistinguishable.

e) How many permutations contain the substring "RANT"?

**Solution**

Treat "RANT" as a super-letter. The letters we have left to permute are 1 H, 2 Is, 3 Ls, 1 Y, 1 C, 1 O and 1 N. Thus we have 11 letters total of which 1 appears twice and another one appears 3 times. Just use the regular permutation with repetition formula to arrive at  $\frac{11!}{2!3!}$  permutations with the substring RANT. (Note, this technique would not work if there were 2 Rs, 2 As, 2 Ns and 2 Ts in the original set of letters to permute.)

3) A class contains 25 girls and 22 boys. For all parts of this question, each boy and girl are distinguishable from one another. Answer the following questions:

a) In how many ways can a committee of one boy and one girl be chosen?

**Solution**

Since we want exactly one boy and one girl,  $\binom{25}{1}\binom{22}{1} = 25 * 22 = 550$  ways.

b) In how many ways can a committee of five students be chosen?

**Solution**

We want a committee of students of either gender. There are a total of 47 students, and no one student can be chosen more than once, so we have  $\binom{47}{5} = 1,533,939$  ways.

c) In how many ways can a committee of four girls and three boys be chosen?

**Solution**

We must first choose 4 girls from the possible 25, and then choose 3 boys from the possible 22. Since for each choice of the 4 girls we have to choose 3 boys, we multiple these values.

So we have  $\binom{25}{4}\binom{22}{3}$  ways.

d) In how many ways can a committee of six students be chosen such that all the students on the committee are the same sex?

**Solution**

We can choose 6 students from the set of 25 girls or 6 students from the set of 22 boys, so we must calculate these numbers individually. We have  $\binom{25}{6} + \binom{22}{6}$  ways.

e) In how many ways can the girls and boys form a line where no two boys are standing next to one another?

**Solution**

Let  $g$  be the number of girls, and  $b$  be the number of boys.

Imagine lining up all of the girls (G's), leaving a placeholder (|) for a boy to potentially be placed:

$$|G_1|G_2|\dots |G_g|$$

Placing  $b$  boys in the line such that no two are standing next to each other amounts to placing these boys in these placeholders, where no one placeholder can be chosen more than once. There are  $\binom{g+1}{b}$  such choices, as there are  $g+1$  placeholders.

Now we must find all of the arrangements of the girls and boys.

Since every person is distinguishable, there are  $25!$  rearrangements of the girls, and  $22!$  rearrangements of boys. Thus the total of total lineups that satisfy the condition that no two boys are standing next to each other is  $\binom{26}{22} * 25! * 22!$

f) How many committees of seven students contain at least two girls?

**Solution**

There are  $\binom{47}{7}$  possible committees. Of these, we do not want to count those committees that have no girls and exactly one girl in them.

There are  $\binom{22}{7}$  committees with no girls and all boys.

There are  $\binom{22}{6}\binom{25}{1}$  committees with six boys and exactly one girl.

Thus, the final result is

$$\binom{47}{7} - \binom{22}{7} - \binom{22}{6}\binom{25}{1}$$

4) How many solutions does the equation  $a + b + c + d + e + f = 25$  have if each variable must be a non-negative integer and  $a \leq 4$ ,  $b \leq 6$  and  $d \geq 5$ ?

**Solution**

Think about this question as distributing 25 non-distinct wooden blocks among 6 children – Anna, Bob, Colin, Dan, Emma, and Fay. The children are sitting in a line with 5 separators (denoting the 5 plus signs here) separating them.

If we were not to have any constraints such as  $a \leq 4$ ,  $b \leq 6$  and  $d \geq 5$ , then each of them can get  $\geq 0$  number of wooden blocks. Assuming that we can tell the difference between the 6 children, means that each of the 6 variables  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  are distinct. We now add to our collection six “artificial” wooden blocks resulting in a total of 31 wooden blocks. We then distribute these 31 wooden blocks to the 6 children. After this distribution is made, we “tax” each of them and make them give back one wooden block. Now it may happen that any one (and perhaps as many as five out of the six) children are left with none. Regardless, we have counted the number of distributions with empty cells allowed ( $\geq 0$  or non-negative integer). The answer in general is  $\binom{m+n-1}{n-1}$  where  $m$  is the number of non-distinct objects (here, wooden blocks) and  $n$  is the number of distinct objects (here, children). Specifically, here, we would have  $\binom{25+6-1}{6-1} = \binom{30}{5}$  ways to distribute the wooden blocks *without any constraints*.

Now, let us get to the actual question where we have constraints.

First, we work with  $\mathbf{d} \geq 5$ . Here the principle is to “set aside” some wooden blocks before the distribution is made. In this case, we give five wooden blocks to Dan (corresponding to variable  $d$ ) in advance. We have  $25 - 5 = 20$  wooden blocks left to distribute among all six children. In this way, Dan is guaranteed to get 5 or more, and the others  $\geq 0$  number of wooden blocks. So then, the answer just with the constraint  $\mathbf{d} \geq 5$  is  $\binom{20+6-1}{6-1} = \binom{25}{5}$ .

Next, let us add the constraint  $\mathbf{a} \leq 4$ . We can set this to be  $\mathbf{a} \geq 5$  and solve just like the way we solve for Dan. Of the remaining 20 wooden blocks (after giving 5 to Dan), let us set aside 5 for Alice. We are left with  $20-5 = 15$  wooden blocks. These 15 have to be distributed among all six children (that way Dan still gets 5 or more). This gives us  $\binom{15+6-1}{6-1} = \binom{20}{5}$  ways to ensure Alice gets five or more. By subtracting  $\binom{20}{5}$  from  $\binom{25}{5}$ , we get the number of ways we can give Dan 5 or more and Alice at most 4 wooden blocks.

Finally, let us add the constraint  $\mathbf{b} \leq 6$ . We can set this to be  $\mathbf{b} \geq 7$  and solve just like the way we solve for Alice. Of the remaining 15 wooden blocks (after giving 5 to Dan and 5 to Alice), let us set aside 7 for Bob. We are left with  $20-7 = 13$  wooden blocks. These 13 have to be distributed among all six children (that way Dan still gets 5 or more). This gives us  $\binom{13+6-1}{6-1} = \binom{18}{5}$  ways to ensure Bob gets 7 or more. Now, by subtracting  $\binom{18}{5}$  from  $\binom{25}{5} - \binom{20}{5}$ , gives us the number of ways we can give Dan 5, Alice at most 4, and Bob at most 6 wooden blocks, but with a little too much subtracted out, which we will address momentarily.

Notice that we have removed solutions for  $\mathbf{a} \geq 5$  and  $\mathbf{b} \geq 7$  twice. We have to add once back. We removed the ways in which  $20 - (5+7) = 8$  wooden blocks could be distributed among the 6 children twice. We now add back  $\binom{8+6-1}{6-1}$  ways back to  $\binom{25}{5} - \binom{20}{5} - \binom{13}{5}$ , giving us

$\binom{25}{5} - \binom{20}{5} - \binom{18}{5} + \binom{13}{5} = 37626$  ways of distribution the wooden blocks, or 37626 solutions to the equation.

5) How many solutions does the equation  $a + b + c + d + e + f + g \leq 30$  have if each variable must be a non-negative integer?

**Solution**

This equation can be reduced to

$$\mathbf{a + b + c + d + e + f + g + h = 30}$$

where the variable  $h$  picks up the slack.

Just like in the solution to question 4, we now have 30 non-distinct wooden blocks to be distributed among 8 distinct children. Using  $\binom{m+n-1}{n-1}$  where  $m$  is the number of non-distinct objects (here, wooden blocks) and  $n$  is the number of distinct objects (here, children), we have

$$\binom{30+8-1}{8-1} = \binom{37}{7} = 10,295,472 \text{ solutions.}$$

6) There are  $N$  users and  $M$  servers with  $M \geq N$ . Each user can send a request to any of the servers. Determine the number of situations in which at least one collision occurs, i.e., there is at least one pair of users that send the request to the same server.

**Solution**

We want the number of ways such that there is at least one conflict.

This value is equivalent to the number of ways where there was no conflict subtracted from total number of ways for requests to be sent out.

The total number of ways for  $n$  people to send a single request to one of  $m$  servers is  $m^n$ , since each of the  $n$  people can choose any of the  $m$  servers.

The number of ways for  $n$  people to send a single request to one of  $m$  servers where there are no collisions would be  ${}_m P_n$ , since this is the same as finding an arrangement of length  $n$  of  $m$  servers.

So, the number of ways for  $m$  people to send requests to  $n$  servers where there was at least one conflict is

$$m^n - {}_m P_n$$

7) How many integers in between 1 and  $10^7$  are divisible by 2, 5 or 11?

**Solution**

We will use the inclusion exclusion principle.

There are three steps to finding the solution.

In the first step, let us find the number of integers divisible by 2, 5, and 11, without worrying about overlaps. Use the floor values on division.

$$\begin{aligned} \text{Number of integers divisible by 2} &= 10000000/2 = 5000000 \\ \text{Number of integers divisible by 5} &= 10000000/5 = 2000000 \\ \text{Number of integers divisible by 11} &= 10000000/11 = 909090 \\ \text{Total: } &7909090 \end{aligned}$$

In the second step, let us count the integers we counted twice. Specifically, we counted the multiples of 10 ( $2*5$ ), 55 ( $5*11$ ), and 22 ( $2*11$ ) twice. We have to remove these counts. Use the floor values on division.

$$\begin{aligned} \text{Number of integers divisible by 10} &= 10000000/10 = 1000000 \\ \text{Number of integers divisible by 55} &= 10000000/55 = 181818 \\ \text{Number of integers divisible by 22} &= 10000000/22 = 454545 \\ \text{Total: } &1636363 \end{aligned}$$

Notice that in the process of removing the multiples of 10, 55, and 11, we removed the multiples of 110 ( $2 \times 5 \times 11$ ). We add this count back.

$$\text{Number of integers divisible by 110} = 10000000/110 = 90909$$

The final answer then is  $7909090 - 1636363 + 90909 = \mathbf{6363636}$

8) How many permutations of 3 As, 5 Bs and 7 Cs don't have any consecutive Bs in them?

**Solution**

Use the As and Cs as separators:  $\_ A \_ A \_ A \_ C \_ C \_ C \_ C \_ C \_ C \_ C \_ C \_$ . We can choose any 5 of the 11  $\_$  slots for the Bs in  $\binom{11}{5}$  ways. Then, for each of these selections, we can permute the As and Cs in  $\frac{10!}{3!7!}$  ways for a total of  $\binom{11}{5} \frac{10!}{3!7!}$  permutations without consecutive Bs.

9) Consider the following problem and proposed solution:

Problem: How many permutations of AABBCCDDEEFFGG do not contain any consecutive letters.

Solution: let  $f(n)$  be the solution for  $n$  pairs of matching letters. We can choose the first letter in the permutation to be any one of the  $n$  letters. Then, we can select any of  $2n-2$  slots for the second copy of that letter - anything BUT the second slot. What remains is to recursively fill in  $2(n-1)$  slots with the remaining letters, which can be done in  $f(n-1)$  ways. Thus, we've derived the equation:

$$f(n) = n(2n - 2)f(n - 1)$$

We can quickly note that  $f(2) = 2$  since the two valid arrangements of AABB are ABAB and BABA.

Now, we can plug in for  $n = 7$  to solve the given query:

$$\begin{aligned} f(3) &= 3 \times 4 \times f(2) = 12 \times 2 = 24 \\ f(4) &= 4 \times 6 \times f(3) = 24 \times 24 = 576 \\ f(5) &= 5 \times 8 \times f(4) = 40 \times 576 = 23040 \\ f(6) &= 6 \times 10 \times f(5) = 60 \times 23040 = 1382400 \\ f(7) &= 7 \times 12 \times f(6) = 84 \times 1382400 = 116121600 \end{aligned}$$

Thus, there are 116,121,600 permutations of AABBCCDDEEFFGG without 2 of the same letter appearing consecutively.

What is the flaw in this approach? Conceptually, is this approach overcounting or undercounting?

**Solution**

This approach is undercounting. Let us look at an example for  $f(4)$ .

The reason the formula under-counts is that while it considers placing every letter first and placing its pair in different slots, so for example, we have the following molds:

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A _ A _ _ _ _
A _ _ A _ _ _
A _ _ _ A _ _
A _ _ _ _ A _
A _ _ _ _ _ A
A _ _ _ _ _ _ A

```

In building our new solutions, we only build them off of old solutions. Thus, we'd never build off the string BBCDCD because this isn't a valid solution of length 3. **BUT**, consider placing this substring in the locations denoted by the first mold:

A B A B C D C D

So, the problem is that in this method of accounting, when we place the same letter in slots 1 and 3, we **ARE** allowed to place the same letters in slots 2 and 4, but this method disallows that because it **ONLY** inserts valid solutions of length  $2(k-1)$  when building a valid solution of length  $2k$ . If you carefully notice, the same issue occurs in **EVERY** mold except for the last one where the whole string of length  $2(k-1)$  is sandwiched between the first and last letter that are the same.

10) How many positive integer solutions does the equation  $a + b + c = 100$  have if we require  $a < b < c$ ?

**Solution**

Since the integers for this question are all positive, add 1 to the values of a, b and c and now change them to be non-negative, adding to 97. So, now, we want the number of non-negative integer solutions to

$$a' + b' + c' = 97$$

with  $a' < b' < c'$ .

Let's get rid of the ordering restriction temporarily. Now, there are  $\binom{97 + 2}{2}$  total non-negative integer solutions. We place these solutions in one of two categories:

- (1) Solutions with 3 distinct integers.
- (2) Solutions with 2 distinct integers.

Notice that in group (1), we count each combination we want to count precisely 6 times. In group (2) we count each combination that we want to count precisely 3 times. This is handy because we'll be able to use the division principle after subtracting out the items from group (2).

Solutions with 2 distinct integers must have a repeated the integer. The repeated integer can range from 0 to 48. Note that 49 is too high since  $49 + 49 = 98 > 97$  (our total). Thus, there are 49 unordered sets of 3 integers with a repeated integer that add to precisely 97. These sets range from  $\{0,0,97\}$ ,  $\{1,1,95\}$ , to  $\{48,48,1\}$ . Each of these sets can be ordered in precisely 3 ways. Thus, there are  $3 \times 49 = 147$  solutions to our original problem that fall into category (2) that we DON'T want to count. It follows that the remaining  $\binom{97+2}{2} - 147$  solutions are all of the variety described in group (1), with 3 distinct integers. Our count counts each of the solutions we actually want 6 times. (Essentially, the solution  $a=1, b=2, c=94$  is counted along with  $(a=1, b=94, c=2)$ ,  $(a=2, b=1, c=94)$ ,  $(a=2, b=94, c=1)$ ,  $(a=94, b=1, c=2)$  and  $(a=94, b=2, c=1)$ ). Thus, to get our final answer, just divide this value by 6:

$$\frac{\binom{99}{2} - 147}{6} = \frac{4851 - 147}{6} = \frac{4704}{6} = \mathbf{784}$$

**Alternate Solution (by brute force)**

Consider setting  $a = 1, 2, 3, \dots, 32$ .

When  $a = 1$ , we can have  $b = 2$  through  $b = 49$ , for 48 solutions.

When  $a = 2$ , we can have  $b = 3$  through  $b = 48$ , for 46 solutions.

When  $a = 3$ , we can have  $b = 4$  through  $b = 48$  for 45 solutions.

When  $a = 4$ , we can have  $b = 5$  through  $b = 47$  for 43 solutions.

In general, when  $a = 2k$  for some integer  $k$ , we can have  $b = 2k+1$  through  $b = 50 - k - 1$ , for a total of  $50 - 3k - 1$  solutions.

In general, when  $a = 2k+1$  for some integer  $k$ , we can have  $b = 2k+2$  through  $b = 50 - k - 1$ , for a total of  $50 - 3k - 2$  solutions.

The largest value for  $a$  is  $a = 32$ , so we range from  $k = 1$  to 16 for even values of  $a$ , and  $k = 0$  to 15 for odd values of  $k$ . Our final total is:

$$\sum_{k=1}^{16} (50 - 3k - 1) + \sum_{k=0}^{15} (50 - 3k - 2) =$$

$$16 \times 50 - 3 \times \frac{16 \times 17}{2} - 16 + 16 \times 50 - 3 \times \frac{15 \times 16}{2} - 32 = \mathbf{784}$$