Fall 2016 COT 3100 Section 1 Homework 4 Solutions
Assigned: 10/4/2016
Due: 10/14/2016

Note: For all of these questions, please solve without a calculator or computer. Use the methods shown in class to solve these problems manually. You may double check basic arithmetic with a calculator.

1) What is the sum of an arithmetic sequence with 100 terms with the first term equal to 17 and a common difference of 4?

Solution
Each term can be expressed as 17 + 4(i - 1), or 13 + 4i

$$\sum_{n=1}^{100}(13+4i) = \sum_{n=1}^{100} 13 + 4\sum_{n=1}^{100} i = 1300 + 4\left(\frac{100*101}{2}\right) = 1300 + 20200 = 21500$$

Alternatively, we have $a_{100} = a_1 + 99d = 17 + 99(4) = 413$.

$$S_{100} = \frac{100(a_1 + a_{100})}{2} = 50(17+413) = 21500$$

2) Let $a_1, a_2, \ldots, a_{65}$ be an arithmetic sequence such that $a_{22} = 46$ and $a_{35} = 267$. Find $\sum_{i=1}^{65} a_i$.

Solution
Let d be the common difference of the sequence, then we have:

$a_{35} = a_{22} + 13d$
$267 = 46 + 13d$
Thus $d = 17$

$a_1 = a_{22} - 21(17)$
$a_1 = 46 - 357 = -311$
Thus $a_i = -311 + 17(i - 1) = -328 + 17i$

$$\sum_{i=1}^{65}(-328+17i) = \sum_{i=1}^{65} -328 + 17\sum_{i=1}^{65} i = -328*65 + 17*\frac{(65*66)}{2} = 15145$$

3) What is the sum of an infinite geometric sequence with a first term of 7 and a common ratio of $\frac{2}{5}$?

Solution

$$\sum_{i=0}^{\infty} 7 * \left(\frac{2}{5}\right)^n = 7 \sum_{i=0}^{\infty} \left(\frac{2}{5}\right)^n = 7 * \left(\frac{1}{1-\left(\frac{2}{5}\right)}\right) = \frac{7}{\frac{3}{5}} = \frac{35}{3}$$

4) Consider a geometric sequence $a_1, a_2, \ldots, a_{15}$ such that $a_5 = 48$ and $a_9 = 768$. Find $\sum_{i=1}^{15} a_i$.

Solution

$a_9 = a_5 * r^4$
$768 = 48 * r^4$
$16 = r^4$
$\pm 2 = r$

$a_5 = a_1 * 2^4$
$48 = a_1 * 16$
$a_1 = 3$
$a_i = 3 * 2^{i-1}$ or $a_i = 3 * (-2)^{i-1}$

So one possibility for the sum is:

$$\sum_{i=1}^{15} 3 * 2^{i-1} = 3 \sum_{i=1}^{15} 2^{i-1} = 3 \sum_{i=0}^{14} 2^i = 3 * \frac{(1-2^{15})}{1-(2)} = 3 * 36767 = 98301$$

The other possibility for the sum is:

$$\sum_{i=1}^{15} 3 * (-2)^{i-1} = 3 \sum_{i=1}^{15} (-2)^{i-1} = 3 \sum_{i=0}^{14} (-2)^i = 3 * \frac{(1+2^{15})}{1-(-2)} = 36769$$

5) Determine the following sum in terms of n: $\sum_{i=1}^{n}(i(i+1)(i+2))$.

Solution

$i(i+1)(i+2) = (i^2+i)(i+2) = i^3+2i^2+i^2+2i = i^3+3i^2+2i$

$\sum(i^3+3i^2+2i) = \sum i^3 + 3\sum i^2 + 2*\sum i$

$\frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$

$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{2} + n(n+1)$

$= \frac{n(n+1)}{4}[n(n+1)+2(2n+1)+4]$

$= \frac{n(n+1)}{4}[n^2+n+4n+2+4]$

$= \frac{n(n+1)}{4}[n^2+5n+6]$

$= \frac{n(n+1)(n+2)(n+3)}{4}$

6) By noticing that $\sum_{i=1}^{n}(2i+1) = \sum_{i=1}^{n}((i+1)^2-i^2) = \sum_{i=1}^{n}(i+1)^2 - \sum_{i=1}^{n} i^2$, and noticing the telescoping nature of the sum on the right, determine $\sum_{i=1}^{n}(2i+1)$.

Solution

$$\sum_{i=1}^{n}(2i+1) = \sum_{i=1}^{n}((i+1)^2-i^2) = \sum_{i=1}^{n}(i+1)^2 - \sum_{i=1}^{n} i^2$$

$$= \sum_{i=2}^{n} i^2 + (n+1)^2 - 1 - \sum_{i=2}^{n} i^2$$

$$= (n+1)^2 - 1$$

$$= n^2+2n+1-1$$

$$= n^2+2n$$

More visually, we have:

$$\sum_{i=1}^{n}(2i+1) = \sum_{i=1}^{n}((i+1)^2-i^2) = \sum_{i=1}^{n}(i+1)^2 - \sum_{i=1}^{n} i^2$$

$$= \mathbf{2^2} + \mathbf{3^2} + \mathbf{4^2} + \cdots + \boldsymbol{n^2} + (n+1)^2$$

$$-(1^2 + \mathbf{2^2} + \mathbf{3^2} + \mathbf{4^2} + \cdots + \boldsymbol{n^2})$$

$$= (n+1)^2 - 1^2 = n^2+2n+1^2-1^2 = n^2+2n$$

7) What is $\sum_{i=1}^{\propto} i(\frac{3}{5})_{i-1}$?

Solution

$\sum_{k=1}^{\infty} k(x)^{k-1} = \frac{1}{(1-x)^2}$

$k = i, x = \frac{3}{5}$

$\frac{1}{\left(1-\frac{3}{5}\right)^2} = \frac{1}{\left(\frac{2}{5}\right)^2} = \frac{1}{\left(\frac{4}{25}\right)} = \frac{25}{4}$

8) What is $\begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 4 & -3 \end{bmatrix}$?

Solution

$\begin{bmatrix} 1 & 3 & 4 \\ 6 & 2 & -3 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 4 & -3 \end{bmatrix}$

$= \begin{bmatrix} (1*2)+(3*-1)+(4*4) & (1*3)+(3*1)+(4*-3) \\ (6*2)+(2*-1)+(-3*4) & (6*3)+(2*1)+(-3*-3) \end{bmatrix}$

=$\begin{bmatrix} 15 & -6 \\ -2 & 29 \end{bmatrix}$

9) Calculate the (a) join of M_1 and M_2, (b) meet of M_1 and M_2, and (c) the Boolean product of M_1 and M_3.

$$m_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, m_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, m_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution

(a)

$$\begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 1 & 1 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

10) Express the system of equations below as a matrix multiplication:

3x + 4y + 5z = 16
2x - 5y + 11z = 3
x + 6y + 2z = 15

Solution

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -5 & 11 \\ 1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 3 \\ 15 \end{bmatrix}$$

11) Using mathematical induction, prove that $\sum_{i=1}^{n}\big(i(i!)\big) = (n+1)! - 1$, for all positive integers n.

Solution

Base Case: n = 1
$LHS: \sum_{i=1}^{1}\big(i(i!)\big) = 1$
$RHS: (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$
$1 = 1$, so this case is true.

Inductive Hypothesis: Assume that for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k}\big(i(i!)\big) = (k+1)! - 1$$

Inductive Step: Prove for n = k + 1 that $\sum_{i=1}^{k+1}\big(i(i!)\big) = (k+2)! - 1$

$\sum_{i=1}^{k+1}\big(i(i!)\big)$
$= \sum_{i=1}^{k}\big(i(i!)\big) + \sum_{i=k+1}^{k+1}\big(i(i!)\big)$ By summation rules
$= (k+1)! - 1 + (k+1)(k+1)!$ By I.H, definition of summation
$= (k+1)(k+1)! + (k+1)! - 1$ By Transitive Property of addition
$= \big((k+1)!\big)(k+1+1) - 1$ By factoring out k+1
$= (k+1)!\,(k+2) - 1$ Simplification
$= (k+2)! - 1$ Definition of Factorial
$= ((k+1)+1)! - 1$

Thus, since we proved the statement is true for both n=1 and all following n, the statement holds for all positive n.

12) Note that the n^{th} Harmonic number, denoted H_n, equals $\sum_{i=1}^{n} \frac{1}{i}$. Using mathematical induction, prove that $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, for all positive integers n.

Solution

Base Case: n=1

$LHS: \sum_{i=1}^{1} H_i = H_1 = 1$

$RHS: (1+1)H_1 - 1 = 2 * 1 - 1 = 1$

$1 = 1$

So, the statement is proved for n = 1.

Inductive Hypothesis: Assume that for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k} H_i = (k+1)H_k - k$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1} H_i = (k+2)H_{k+1} - (k+1)$$

$(k+2)H_{k+1} - (k+1)$

$= (k+2)\left(H_k + \frac{1}{k+1}\right) - k - 1$ Definition of Harmonic

$= kH_k + 2H_k + \frac{k+2}{k+1} - k - 1$ Distribution

$= kH_k + H_k + H_k + \frac{k+2}{k+1} - k - 1$ $2x = x + x$

$= (k+1)H_k - k + H_k + \frac{k+2}{k+1} - 1$ Communitive, factor out H_k

$= (\sum_{i=1}^{k} H_i) + H_k + \frac{k+2}{k+1} - \frac{k+1}{k+1}$ Inductive Hypothesis, $1 = \frac{k+1}{k+1}$

$= \left(\sum_{i=1}^{k} H_i\right) + H_k + \frac{1}{k+1}$ Subtraction

$= \left(\sum_{i=1}^{k} H_i\right) + H_{k+1}$ Definition of Harmonic Number

$= \sum_{i=1}^{k+1} H_i$ Summation Rules

Thus we have proved $\sum_{i=1}^{k+1} H_i = (k+2)H_{k+1} - (k+1)$. Since we have proved the base case, along with the inductive case, the statement is true for all positive n.

13) Using mathematical induction, prove that 21 | ($4^{n+1} + 5^{2n-1}$) for all positive integers n.

Solution

Base Case: n = 1

$4^2 + 5^1 = 21.$

$21|21$ is true.

Inductive Hypothesis: assume for positive integer n = k that $21|$ ($4^{k+1} + 5^{2k-1}$).

Thus $(4^{k+1} + 5^{2k-1}) = 21 * c$, for some integer c.

In addition, $4^{k+1} = 21c - 5^{2k-1}$

Inductive step: Prove for n = k + 1 that $21|$ ($4^{k+1} + 5^{2(k+1)-1}$).

$4^{k+2} + 5^{2(k+1)-1}$	
$=4^{k+2} + 5^{2k+1}$	Simplification
$= 4 * 4^{k+1} + 5^2 * 5^{2k-1}$	Exponent Multiplication
$= 4(21c - 5^{2k-1}) + 5^2 * 5^{2k-1}$	Inductive Hypothesis
$= (21(4)c - (4)\, 5^{2k-1}) + 25 * 5^{2k-1}$	Simplification
$= 21(4)c + 5^{2k-1}(25 - 4)$	Factor out 5^{2k-1}
$= 21(4)c + 5^{2k-1}(21)$	Subtraction
$= 21(4c + 5^{2k-1})$	Factor out 21
$= 21c'$	$(4c + 5^{2k-1})$ is an integer
$21\|21c'$	Definition of Divisibility

Thus, $21|4^{k+2} + 5^{2(k+1)-1}$, proving the inductive step. Since the base case and inductive step was proved, the statement holds.

14) Using mathematical induction, prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all positive integers n.

Solution

Base Case: n = 1

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Inductive Hypothesis: Assume that for an arbitrary positive integer n = k that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Inductive step: Prove for n = k+1 that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Definition of Matrix Exponentiaiton

$= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Inductive Hypothesis

$= \begin{bmatrix} 1*1+k*0 & 1*1+k*1 \\ 0*1+1*0 & 0*1+1*1 \end{bmatrix}$ Definition of Matrix Multiplication

$= \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$ Simplification

Thus we have proved the inductive case. Since both base case and inductive case was proven, the statement is true on all positive integers.

15) Using mathematical induction, prove that $\sum_{i=1}^{n} i^2 < n^3$, for all positive integers n ≥ 2. (Please do this proof as written instead of proving the equality and then arguing that the real formula is less than n^3. My goal here is to have you practice the mechanics of using induction on an inequality.)

Solution

Base Case: n = 2

$LHS: \sum_{i=1}^{2} i^2 = 1^2 + 2^2 = 5$

$RHS: 2^3 = 8$

5 < 8, proving the base case.

Inductive hypothesis: Assume for an arbitrary integer n = k, k ≥ 2 that $\sum_{i=1}^{k} i^2 < k^3$.

Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} i^2 < (k+1)^3 = k^3 + 3k^2 + 3k + 1$

Inductive Case:

$\sum_{i=1}^{k} i^2 < k^3$ From Inductive Hypothesis

$\sum_{i=1}^{k} i^2 + (k+1)^2 < k^3 + (k+1)^2$ Add $(k+1)^2$ to both sides

$\sum_{i=1}^{k+1} i^2 < k^3 + (k+1)^2$ Definition of Sums

Note $k^3 + (k+1)^2 = k^3 + k^2 + 2k + 1$, and $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.

$k^3 + k^2 + 2k + 1 < (k+1)^3$ $k > 0$

$\sum_{i=1}^{k+1} i^2 < (k+1)^3$ Transitivity of Inequalities.

Thus, the inductive case has been proven. Since both the base case and the inductive case has been proven, the statement holds.

Note: A more traditional way to prove this is as follows:

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\ &< k^3 + k^2 + 2k + 1 \\ &< k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3\end{aligned}$$