Fall 2016 COT 3100 Section 1 Homework 2 Solutions
Assigned: 8/30/2016
Due: 9/9/2016

1) From the premises shown below, prove the conclusion shown. Afterwards, plug in phrases for p, q, r and s that correspond to a natural argument.

$$p \to q$$
$$q \to r$$
$$(q \land r) \to s$$
$$p$$

$$s$$

Solution

1. $p \to q$ Premise
2. p Premise
3. q Using Modus Ponens on 1 and 2.
4. $q \to r$ Premise
5. r Using Modus Ponens on 3 and 4.
6. $q \land r$ Applying Rule of Conjunction to 3 and 5.
7. $(q \land r) \to s$ Premise
8. s Using Modus Ponens on 6 and 7.

2) Consider a proof that begins, "Let n be an arbitrary positive integer that isn't a perfect square", and ends, "Thus, n has an even number of factors." Which of the four following rules of inference for quantifiers is such a proof using: universal instantiation, universal generalization, existential instantiation, or existential generalization?

Solution

Universal Generalization - this is the rule that allows us to take an arbitrary element from a set to prove a "for all" statement about items from that set. This is critical since it would be impossible for us to go through each unique positive integer that isn't a perfect square and count the factors of each.

3) Using proof by cases, show that for all integers n, $n(n+1)$ is even.

Solution

Let us first enumerate the number of cases for this proof. An integer n can be either even or odd. We, therefore, have two cases to prove.

Case 1. When n is even, n may be written as $n = 2*r$, for some integer r, then

$n(n+1) = 2*r(2*r+1) = 2(r^2 + r)$

Since r is an integer, it follows that $(r^2 + r)$ is as well, thus the original quantity is even (since we've expressed it as 2 times an integer.)

Case 2. When *n* is odd, *n* may be written as $n = 2*r + 1$, for some integer *r* then,

$$n(n+1) = (2r+1)(2r+1+1)$$
$$= (2r+1)(2r+2)$$
$$=(2r+1)(r+1)2$$

Since *r* is an integer, it follows that $(2r+1)(r+1)$ is as well, thus the original quantity is even.

4) Using proof by contradiction, show that for all integers $\boldsymbol{n}$, $\boldsymbol{n(n+1)}$ is even.

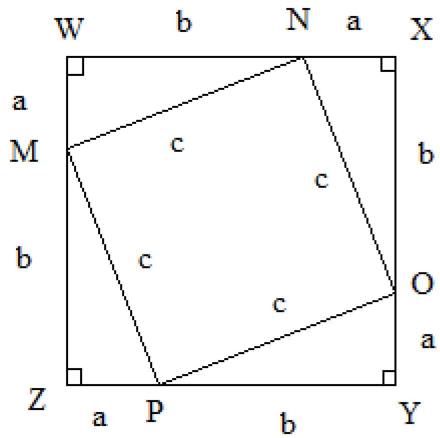
Solution

The first step in building a proof by contradiction is to assume the opposite. Accordingly, let it be the case that *n(n+1)* is odd, for all integers *n*.

We know that a product of two integers is odd when both of the integers are odd. This means that both *n* and *n+1* are odd. It is, however, not possible for both *n* and *n+1* to be odd, which contradicts the premise. To see why *n* and *n+1* can't be odd, simply note that distinct odd numbers must differ by at least 2. (A level of detail not necessary for this response is as follows: any two distinct odd numbers can be represented as 2a + 1 and 2b+1, for distinct integers a and b. The difference without regard to sign between these two is $2|a - b|$. Since $a - b \neq 0$, and both are integers, the absolute value of their difference is at least one, so $2|a - b| \geq 2$.) Therefore, it is not possible for both *n* and *n+1* to be odd, since they are distinct integers with a difference of 1.

Since it is not possible for both *n* and *n+1* to be odd, our assumption that *n(n+1)* is odd is incorrect. Therefore, it must be the case the *n(n+1)* is even.

5) Using the diagram shown below, use direct proof to prove the Pythagorean Theorem. (Hint: describe the area of the full figure in two different ways.) In the diagram, the vertices WXYZ determine a square with vertices M, N, O, P on the sides WZ, WX, XY, and YZ, respectively. The triangle MWN is a right triangle with side lengths *a* and *b*, and hypotenuse *c*. Three other triangles congruent to this one appear in the figure.



Solution

The Pythagorean Theorem states that in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. If we let, c denote the hypotenuse, and a and b to denote the other two sides, Pythagorean Theorem may be expressed as $c^2 = a^2 + b^2$.

The figure can be considered as a square, enclosed by the sides WX, XY, YZ, and ZW, or as composed of four congruent right angled triangles and a square. Since these are the interpretations of the same entity, their areas have to be the same.

The area of the square WXYZ with length of a side equal to $(a+b)$ is $(a+b)^2$. Let us denote this as (1).

Next, the sum of the areas of the four congruent triangles with length of base a and height b, and the area of the square MNOP with length of a side equal to c is

$$4 * ½ * a * b + c^2 = 2ab + c^2$$

Let us denote this as (2).

Since (1) and (2) measure the area of the same entity, $(a+b)^2 = 2ab + c^2$

Expanding $(a+b)^2$, $a^2 + b^2 + 2ab = 2ab + c^2$

Subtracting $2ab$ on both sides, we obtain $a^2 + b^2 = c^2$, or the Pythagoras Theorem.

6) **Google the terms "geometric mean" and "harmonic mean" to find their definitions. Prove that the geometric mean of two real numbers *a* and *b*, is greater than or equal to the harmonic mean of *a* and *b*. (Note: The question should read "two positive real numbers". A harmonic mean with 0 is undefined.)**

Solution: Let us begin with the definitions.

The "geometric mean", G, is defined as the n^{th} root of the product of *n* numbers.

For two real numbers *a* and *b*, their geometric mean is $\sqrt[2]{ab}$.

The "harmonic mean", H, is defined as reciprocal of the arithmetic mean of the reciprocals. This leads us to also use the definition of the *arithmetic mean*. The "arithmetic mean", A, of *n* numbers is the sum of the *n* numbers divided by *n*. For two real numbers *a* and *b*, their arithmetic mean is $\frac{(a+b)}{2}$.

For two real numbers *a* and *b*, their reciprocals are $\frac{1}{a}$ and $\frac{1}{b}$. The arithmetic mean of the reciprocals is $\frac{(\frac{1}{a}+\frac{1}{b})}{2}$. Finally, the harmonic mean which is the reciprocal of the arithmetic mean of the reciprocals is $\frac{2ab}{a+b}$.

In the case of the two numbers *a* and *b*, the harmonic mean, H, can be expressed as H = $\frac{G^2}{A}$. By the inequality of arithmetic and geometric means,

$$\frac{(a+b)}{2} \geq \sqrt[2]{ab}$$

we know that $\frac{G}{A} \leq 1$.
Therefore, $H \leq G$ or, $G \geq H$.

<u>Alternate Solution</u>
In this solution we won't assume the given information that the arithmetic mean of two positive real numbers is greater than or equal to their geometric mean.

Using the terminology from the previous solution, our goal is to show that $G - H \geq 0$.

$$G(a,b) - H(a,b) = \sqrt{ab} - \frac{2ab}{a+b}$$
$$= \frac{(a+b)\sqrt{ab} - 2ab}{a+b}$$
$$= \frac{\sqrt{ab}(a+b-2\sqrt{ab})}{a+b}$$
$$= \frac{\sqrt{ab}(\sqrt{a}^2 - 2\sqrt{ab} + \sqrt{b}^2)}{a+b}$$

$$= \frac{\sqrt{ab}(\sqrt{a} - \sqrt{b})^2}{a + b}$$
$$\geq 0$$

Since both a and b are positive and the squares of all real numbers are non-negative.

7) Prove the following proposition about arbitrary sets *A*, *B*, *C*, *D* and *E*:

If $A \cup B \subseteq C$ and $C \subseteq D \cap E$, then $A \subseteq D \wedge A \subseteq E$.

<u>Solution</u>
Recall that proofs for the statements in the form of *if-then* can be obtained by following the three steps:

1. Assume that the *if* clause is true
2. Under this assumption, make deductions using rules and laws
3. Show that the *then* clause logically follows from the deductions

Let us apply these.

Step 1. We assume that $A \cup B \subseteq C$ and $C \subseteq D \cap E$ are true. This means

$$\forall x\ [(x \in A \vee x \in B) \Rightarrow x \in C]$$

$$\forall x\ [x \in C \Rightarrow (x \in D \wedge x \in E)\ ]$$

We have to show that $A \subseteq D \wedge A \subseteq E$, or

$$\forall x\ [x \in A \Rightarrow (x \in D \wedge x \in E)]$$

Step 2. Making deductions from the rules and laws known to us.

1.	$\forall x\ [(x \in A \vee x \in B) \Rightarrow x \in C]$	Premise
2.	$\forall x\ [x \in C \Rightarrow (x \in D \wedge x \in E)\ ]$	Premise
3.	$\forall x\ [(x \in A \vee x \in B) \Rightarrow (x \in D \wedge x \in E)]$	Law of Syllogism (1, 2)
4.	$(a \in A \vee a \in B) \Rightarrow (a \in D \wedge a \in E)$	3 and Rule of Universal Specification
5.	$\neg(a \in A \vee a \in B) \vee (a \in D \wedge a \in E)$	Implication Identity on 4
6.	$(\neg(a \in A) \wedge \neg(a \in B)) \vee (a \in D \wedge a \in E)$	DeMorgan's Law on 5
7.	$(\neg(a \in A)) \vee (a \in D \wedge a \in E)$	Rule of Conjunctive Simplification on 6
8.	$a \in A \Rightarrow (a \in D \wedge a \in E)$	Implication Identity on 7
9.	$\forall x\ [x \in A \Rightarrow (x \in D \wedge x \in E)]$	8 and Rule of Universal Generalization

Step 3. We showed that the conclusion, $\forall x\ [x \in A \Rightarrow (x \in D \wedge x \in E)]$, logically follows from the deductions in Step 2.

Alternate Solution
Here is an alternate solution using direct proof. We must show that $A \subseteq D \wedge A \subseteq E$.

Thus, we must show that for all elements x∈A, that x∈D ∧ x∈E. We do this by considering an arbitrary element x such that x∈A.

Since x∈A, by definition of union, it follows that $x \in A \cup B$. Since $A \cup B \subseteq C$, by definition of subset we ascertain that $x \in C$. Since $C \subseteq D \cap E$, by definition of subset we can conclude that $x \in D \cap E$. Finally, by applying the definition of intersection, we deduce separately that $x \in D$ and $x \in E$, showing both that $A \subseteq D \wedge A \subseteq E$.

Note: This solution is logically equivalent to the originally given one, but is written out in a different format. Either format will be fine for this course.

8) Disprove the following proposition about arbitrary sets A, B, C, D and E via counter-example:

If $A \cap B \subseteq C$ and $C \subseteq D \cup E$, then $A \subseteq D \vee A \subseteq E$.

Solution
Note that the question asks for proof using a counter-example. This means that we have to find a set of values, or an example, that disproves the proposition.

The proposition reads as follows,

If the intersection of sets A and B is a subset of C, and C in turn is a subset of the union of the sets D and E, then the set A is a subset of set D or the set A is a subset of set E.

The proposition means,

All the elements that are common between sets A and B belong to the set C. All the elements in set C belong to the set consisting of all the elements from sets D and E. Given these, it follows that all the elements in set A are wholly contained in set D or in set E.

We have to construct an example where sets D and E do not contain all the elements of set A.
Let A = {1, 2, 3} and B = {3, 4, 5}. Then, $A \cap B$ = {3}

Let C = {3, 8, 9} such that $A \cap B \subseteq C$ holds true.

Next, let D = {15, 8, 12} and E = {9, 3, 18} such that $D \cup E$ = {15, 8, 12, 9, 3, 18} contains all the elements of C.

Clearly, neither D contains all the elements of A nor E contains all the elements of A.

We have thus disproved the given proposition by using a counter-example.

9) List out each element in the set $\wp(\emptyset)$. (This is read as the power set of the empty set.) How many elements are in this set?

Solution

The power set of a set is a set that consists of all the possible subsets of the set, including the empty set. Thus, the power set of the empty set is { $\emptyset$ }, the set containing the empty set and nothing else.

10) In class, a Cartesian Product was used to describe a possible set of value meals. Come up with a different analogy to describe a Cartesian Product.

Solution

Consider a non-empty set of t-shirts and a non-empty set of summer shorts (it's Florida!).

T-shirts = {Red, White, Grey}
Shorts = {Black, Blue}

The Cartesian Product = {(Red, Black), (Red, Blue), (White, Black), (White, Blue), (Grey, Black), (Grey, Blue)}

The Cartesian product of these two sets would be all of the outfits one could theoretically wear given that the first set contained a list of all the possible t-shirts one owns and the second set contains all of the shorts one owns. (Though maybe in real life there might be certain elements of the Cartesian product that correspond to outfits you wouldn't actually want to wear.)