SUMMATIONS

COP 3502

Summations

- Why do we need to go over summations?
  - This isn't a math class!
- Many times, analyzing an algorithm to determine its efficiency requires adding up many numbers.
  - This can be represented by a summation

Summations

- For example,
  - If we had the sequence 1+2+3+4+5
  - This can be represented by the following summation:

Stopping condition

$$\sum_{i=1}^{5} i$$

What we're summing

Starting condition

Does this remind you of anything we've seen in code?

```
int sum = 0;
for (i=1; i<= 5; i++)
    sum += i;
```

Summations

If we're given a summation,

$$\sum_{k=2}^{14} 2k + 1$$

Total = 0 +7 +9 +... 29

k = 2 → **2k+1 = 5**

k = 3 → **2k+1 = 7**

k = 4 → **2k+1 = 9**

...

k = 14 → **2k+1 = 29**

We can evaluate it in this way:

1) Create a running total set to 0.
2) Set the variable in the bottom (k) of the sum equal to the initial value given, (2)
3) Plug this value into the expression, (2k+1)
4) Add this to your running "total".
5) If your variable equals the last value listed, (14) stop and your answer is what is stored in total.

   -- Otherwise plug in the next integer value for the variable and go to step 3.

In code we would have this:

```
int total = 0;
for (k=2; k<=14; k++)
    total += (2*k+1);
```

Summations

- In general we would say the following:

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \ldots + f(b)$$

- Let's use our example from before, $\sum_{k=2}^{14} 2k + 1$
  - Where f(k) = 2k + 1

$$\sum_{k=2}^{14} f(k) = f(2) + f(3) + f(4) + \ldots + f(14)$$

$$= 5 + 7 + 9 + \cdots + 29$$

But what if we don't want to add up all these #'s? We can apply our formulas for solving summations…

Summations

Formula 1 – can take out constants

- The first formula we have is for a summation with just a constant.

$$\sum_{k=a}^{b} c$$

  - Notice that c does not change with k,
    - so it's constant
  - With constants we can pull them outside the summation:

$$\sum_{k=a}^{b} c \Rightarrow c\sum_{k=a}^{b} 1$$

Summations

Formula 2 – Summing a constant

$$\sum_{k=a}^{b} c = c\sum_{k=a}^{b} 1 = (b-a+1)c$$

- Let's look at a specific example

$$\sum_{i=3}^{7} 5 = 5\sum_{i=3}^{7} 1 = 5 * (7-3+1)$$

Summations
Formula 3 – Sum of i

- If we look at a more difficult summation
  - (that we saw last time) we can derive the formula for it using a clever trick.

$$\sum_{i=1}^{n} i$$

$S = 1+2+3+4+\ldots+(n-1)+n$

$\underline{+\ S = n+(n-1)+(n-2)+\ldots+2+1}$

$2S = (n+1)+ (n+1)+ (n+1)+\ldots+ (n+1)$

$2S = n(n+1)$

$\underline{\mathbf{S = n(n+1)/2}}$

Summations

- Now let's look at a few quick uses of this formula:

$$\boxed{\sum_{i=1}^{n} i = n(n+1)/2}$$

$$\sum_{k=1}^{100} \mathrm{k} = \text{???}$$

$$\sum_{k=1}^{2n} \mathrm{k} = \text{???}$$

$$\sum_{k=1}^{4n-1} \mathrm{k} = \text{???}$$

Summations

Formula 4 – Splitting up expressions

- You can split up the terms in a summation into separate summations

$$\boxed{\sum_{k=a}^{b} (f(k) + g(k)) = \sum_{k=a}^{b} f(k) + \sum_{k=a}^{b} g(k)}$$

$$\sum_{k=1}^{n} (k+3) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 3 = \frac{n(n+1)}{2} + 3n = \frac{n^2 + 7n}{2}$$

Summations

Formula 5 – Change start to 1

- Sometime summations don’t start from 1 and we need them to to apply our formula
  - So this is what we can do:

$$\sum_{k=20}^{40} f(k)$$

  - In general our formula looks like this:

$$\sum_{k=a}^{b} f(k) = \sum_{k=1}^{b} f(k) - \sum_{k=1}^{a-1} f(k)$$

Summations

- So we now we have all the pieces to solve our original example: $$\sum_{k=2}^{14} 2k+1$$
- Formula 4 – split up the terms:

$$\boxed{\sum_{k=a}^{b} (\mathrm{f(k)} + \mathrm{g(k)}) = \sum_{\mathrm{k=a}}^{\mathrm{b}} \mathrm{f(k)} + \sum_{\mathrm{k=a}}^{\mathrm{b}} \mathrm{g(k)}}$$

$$= \sum_{k=2}^{14} 2k + \sum_{k=2}^{14} 1$$

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$$= \sum_{k=2}^{14} 2k + \sum_{k=2}^{14} 1$$

- Take out the constants:

$$= 2\sum_{k=2}^{14} k + \sum_{k=2}^{14} 1$$

Summations

$$= 2\sum_{k=2}^{14} k + \sum_{k=2}^{14} 1$$

- Formula 1 for the right side: $\boxed{c\sum_{k=a}^{b} 1 = (b-a+1)c}$
  - $\sum_{k=2}^{14} 1$ = 14-2+1 = 13
- And we get: $2\sum_{k=1}^{14} k + 13$

Summations

$$2\sum_{k=2}^{14} k + 13$$

- Formula 4 to change start of left side to 1:

$$\boxed{\sum_{k=20}^{40} f(k) = \sum_{k=1}^{40} f(k) - \sum_{k=1}^{19} f(k)}$$

$$2\left(\sum_{k=1}^{14} k - \sum_{k=1}^{2} k\right) \quad +13$$

Summations

$$2\left(\sum_{k=1}^{14} k - \sum_{k=1}^{2} k\right) \quad +13$$

- Apply Formula 3 to each sum of k:

$$\boxed{\sum_{i=1}^{n} i \;= n(n+1)/2}$$

- 2(14*15/2 – 2*3/2)
  - = 14*15-2*3 = 210

Don't forget about +13!!

- **Final answer = 210 + 13 = 223**

Summations

- **Closed form solutions**
- Not all summations result in a number for an answer.
  - Often the answer has one or more variables in it (usually n for our examples).
  - This is called the **"closed form"** of the summation

Summations

- Examples on the board of finding the closed form of summations.