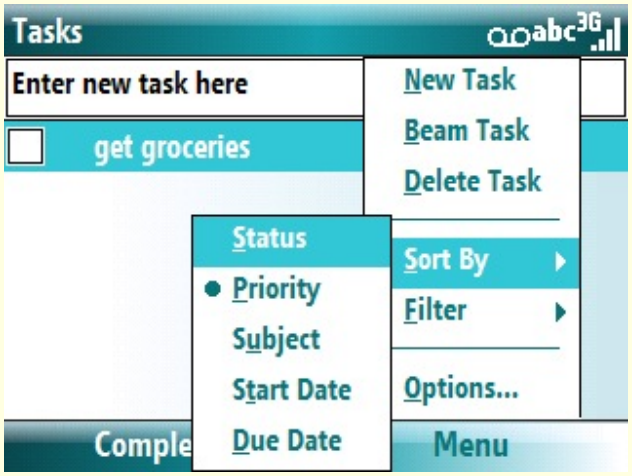
Sorting: $O(n^2)$ Algorithms

Computer Science Department
University of Central Florida

COP 3502 – Computer Science I

Sorting: $O(n^2)$ Algorithms

- Sorting Algorithms:
  - Fundamental problem in Computer Science
  - Sorting is done to make searching easier
  - Most programs do this:
    - Excel, Access, and others.



Sorting: $O(n^2)$ Algorithms

- Sorting Algorithms:
  - We will study several sorting algorithms in this class
    - Some are clearly much faster than others
  - For today, we will go over the "simple sorts"
  - These "simple sorts" all run in $O(n^2)$ time
    - Selection Sort
    - Insertion Sort
    - Bubble Sort

    - We will assume that the input to the algorithm is an array of values (sorted or not)

Sorting: $O(n^2)$ Algorithms

- **Selection Sort:**
  - Given: an array of n unsorted items
  - The algorithm to sort n numbers is as follows:
    1) Find the minimum value in the list of n elements
       - Search from index 0 to index n-1
    2) Swap that minimum value with the value in the first position
       - At index 0
    3) Repeat steps 1 and 2 for the remainder of the list
       - Example:
       - We now start at the 2nd position (index 1).
       - Find minimum value from index 1 to index n-1
       - Swap that minimum value with the value at index 1

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- **Selection Sort:**
  - The algorithm to sort n numbers is as follows:
    - There is a FOR loop that iterates from i = 0 to i = n-1
    - FOR the i^{th} element (as i ranges from 0 to n-1)

    1) Determine the smallest element in the rest of the array
       - To the right of the i^{th} element
    2) Swap the current i^{th} element with the element identified in part (1) above (the smallest element)

    - Essentially:
      - The algorithm first picks the smallest element and swaps it into the first location.
      - Then it picks the next smallest element and swaps it into the next location, etc.

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- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers

20	8	5	10	7
0	1	2	3	4

i = 0

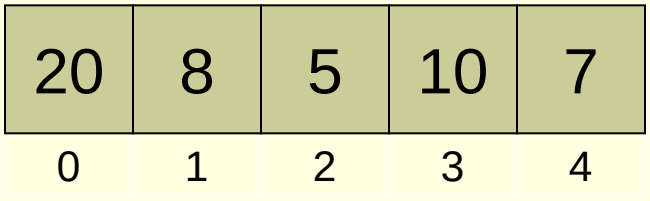
    - Remember, we have a for loop

```
FOR i = 0 to n – 1 {
        Find the minimum value in the range from i to n-1
        SWAP this minimum value with the value at index i
}
```

      - NOTE: i represents the index into the array

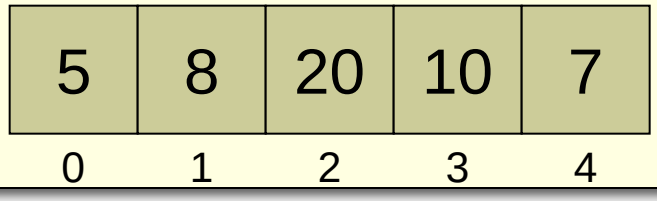
Sorting: $O(n^2)$ Algorithms

- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers



i = 0

    - 5 (at index 2) is the smallest element
      - from the range i = 0 to 4
    - So SWAP the value at index 2 with the value at index 0
      - SWAP the 5 and the 20



Sorting: $O(n^2)$ Algorithms

- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers

5	8	20	10	7
0	1	2	3	4

i = 1

    - 7 (at index 4) is the smallest element
      - from the range i = 1 to 4
    - So SWAP the value at index 4 with the value at index 1
      - SWAP the 7 and the 8

5	7	20	10	8
0	1	2	3	4

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- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers

5	7	20	10	8
0	1	2	3	4

i = 2

    - 8 (at index 4) is the smallest element
      - from the range i = 2 to 4
    - So SWAP the value at index 4 with the value at index 2
      - SWAP the 8 and the 20

5	7	8	10	20
0	1	2	3	4

Sorting: $O(n^2)$ Algorithms

- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers

5	7	8	10	20
0	1	2	3	4

i = 3

    - 10 (at index 3) is the smallest element
      - from the range i = 3 to 4
    - So SWAP the value at index 3 with the value at index 3
      - SWAP the 10 and the 10 (so no swap really happened here)

5	7	8	10	20
0	1	2	3	4

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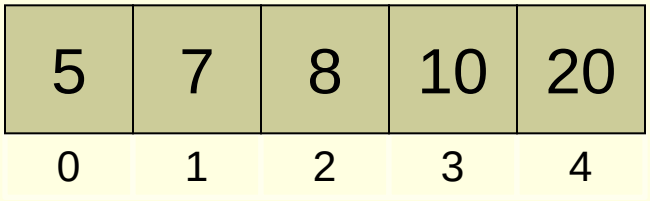
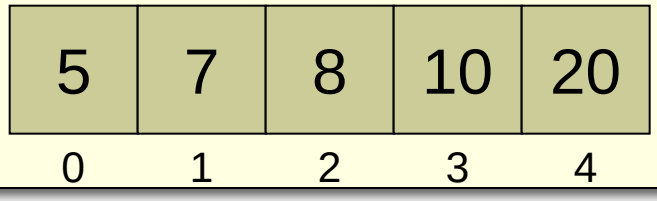
- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers



i = 4

    - 20 (at index 4) is the smallest element
      - from the range i = 4 to 4
    - So SWAP the value at index 4 with the value at index 4
      - SWAP the 20 and the 20 (so no swap really happened here)



Sorting: $O(n^2)$ Algorithms

- **Selection Sort:**
  - Example:
    - Here is an array of 5 integers

5	7	8	10	20
0	1	2	3	4

i = 4

    - The array is now in sorted order
    - We see that the last iteration was not even necessary
      - In code our for loop could look like this:

```
for (i = 0; i < n-1; i++)
```

      - So it won't even iterate on the n-1 step

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- **Selection Sort:**
  - Analysis of Running Time:
    - During the first iteration
      - We “go through” all n items searching for the minimum
      - This is essentially n simple steps
    - During the second iteration, i starts at index 1
      - We “go through” n – 1 items searching for the minimum
      - We do not need to account for the item at index 0
      - Cuz it is already in the correct position!
    - During the third iteration,
      - We “go through” n – 2 items searching for the minimum
      - We do not need to account for the items at index 0 and 1
      - Cuz they are already in correct position

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- **Selection Sort:**
  - Analysis of Running Time:
    - 4^{th} iteration:
      - We will "go through" n – 3 steps
    - 5^{th} iteration
      - We will "go through" n – 4 steps
    - …
    - Final iteration
      - There will simply be one step
    - We can add up the TOTAL number of simple steps
    - TOTAL = n + (n-1) + (n-2) + (n-3) + … + 3 + 2 + 1
    - Is this n^2 steps? Perhaps logn steps? Perhaps n steps?

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- **Selection Sort:**
  - Analysis of Running Time:
    - TOTAL = n + (n-1) + (n-2) + (n-3) + … + 3 + 2 + 1
    - We does this add up to?
      - We need to know this in order to give the Big-O
    - There is a neat trick!
    - Write the equation shown above
    - And then immediately underneath,
      - Write the equation again, but REVERSE the order of the terms
    - Then add the two equations together
      - See what happens
    - Finally, solve for TOTAL

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- **Selection Sort:**
  - Analysis of Running Time:

$$\begin{array}{rl} & \text{TOTAL} = n + (n-1) + (n-2) + (n-3) + \ldots + 3 + 2 + 1 \\ + & \text{TOTAL} = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n \\ \hline & 2*\text{TOTAL} = (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1) \end{array}$$

    - How many terms of (n+1) do we have?
      - We have n of them!
    - So that is n*(n+1)
    - 2*TOTAL = n(n+1)
    - TOTAL = n(n+1)/2
    - So we see that Selection sort runs in $O(n^2)$ time.

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- **Insertion Sort:**
  - This is the sort that most humans apply when sorting documents
  - Example: Playing Cards
    - Players usually keep cards in sorted order
    - When you pick up a new card
      - You make room for the new card and put into its proper place

Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - The card example demonstrates the basic idea of Insertion Sort
    - But the “idea” isn’t exactly the same as sorting an array of items
  - When sorting an array of items, we are ALREADY holding all of the items
  - So how are we “inserting” an item when it is already in the list.
  - We remove the items, one at a time, and then reinsert them into their proper positions

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- **Insertion Sort:**
  - Bookshelf example:
  - If first two books are out of order:
    - Remove second book
    - Slide first book to right
    - Insert removed book into first slot
  - Next, look at third book, if it is out of order:
    - Remove that book
    - Slide 2nd book to right
    - Insert removed book into 2nd slot
  - Recheck first two books again
    - Etc.

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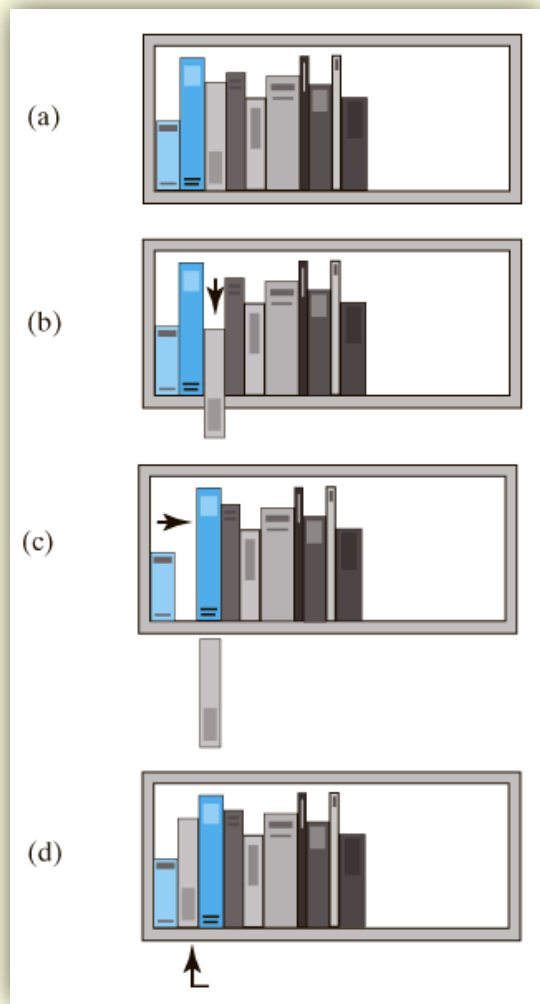
- **Insertion Sort:**
  - Bookshelf example:
    - This picture shows the "insertion" of the third book
      - The 3rd book is removed
      - It is compared with the 2nd book
      - The 2nd book is larger
      - So we slide the 2nd book into the 3rd spot
      - We then compare our original 3rd book with the 1st book
      - They are in order
      - So we simply insert the original 3rd book in the 2nd spot



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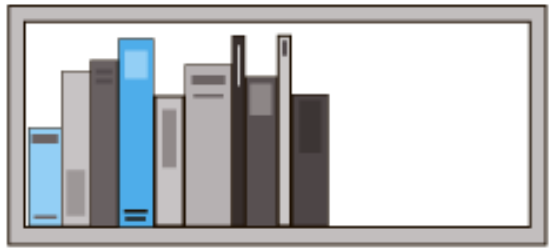
- **Insertion Sort:**
  - Bookshelf example:
    - In general:



1. Remove the next unsorted book.
2. Slide the sorted books to the right one by one until you find the right spot for the removed book.
3. Insert the book into its new position.

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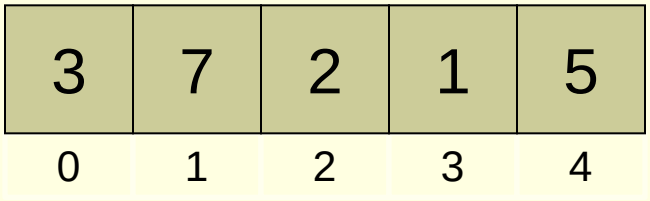
- **Insertion Sort:**
  - Given: an array of n unsorted items
  - The algorithm to sort n numbers is as follows:
    - Starting with the 2nd element,
    - Take each element, one by one, and
    - "Insert" it into a sorted list
    - How do we insert it?
      - continually SWAP it with the previous element until it has found its correct spot in the already sorted list
        - When we say already sorted list, we are referring to the elements to the left of our current element
        - Those elements are already in sorted order

Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - The algorithm to sort n numbers is as follows:
    - For the ith element
      - as i ranges from 1 to n-1 (week skip i = 0, the 1st element)
    - As long as the current element is smaller than the element before it
      - SWAP the two elements
    - Stop when the current element is bigger than the one before it OR there is no element before it
      - Meaning it has reached the front
    - An example should clarify…

Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers



i = 1

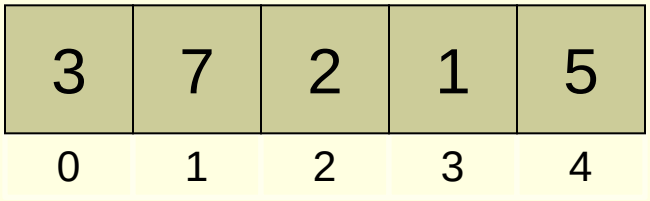
    - Remember, we have a for loop

```
FOR i = 1 to n – 1 {
    WHILE the current element (at index i) is smaller than the
    element before it
        SWAP the two elements
}
```

    - NOTE: i represents the index into the array

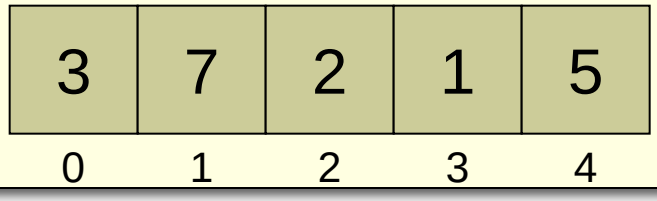
Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers



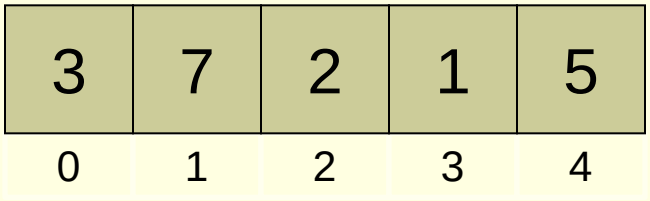
i = 1

    - 7 is the value at index 1
      - Compare 7 to the value at index 0 (which is 3)
    - 7 is greater than 3
      - So there is nothing to swap. Simply re-insert 7 at its place.



Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers

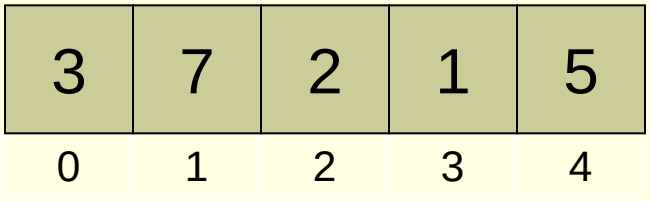


i = 2

    - 2 is the value at index 2
      - Compare 2 to the value at index 1 (which is 7)
    - 2 is smaller than 7
      - So we SWAP
    - BUT we are NOT done!

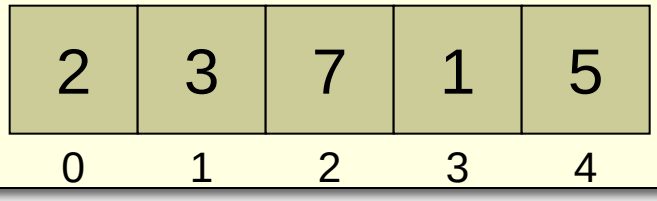
Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers



i = 2

    - We must compare the 2 to the value at index 0
      - which is 3
    - 2 is smaller than 3
      - So we SWAP



Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers

2	3	7	1	5
0	1	2	3	4

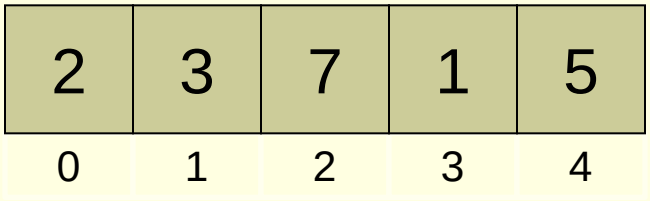
i = 3

    - 1 is the value at index 3
      - Compare 1 to the value at index 2 (which is 7)
    - 1 is smaller than 7
      - So we SWAP
    - 1 is smaller than 3
      - So we SWAP

Continue comparing 1 to the element before it

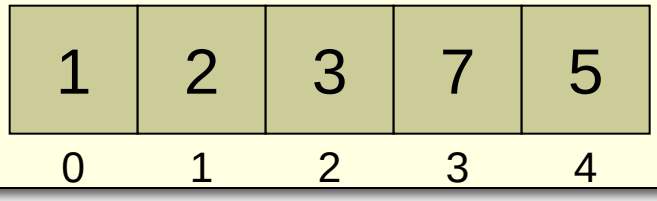
Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers



i = 3

    - 1 is smaller than the value at index 0 (which is 2)
      - So we SWAP
    - There is no element “before” 1 at this point
      - So we simply insert



Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers

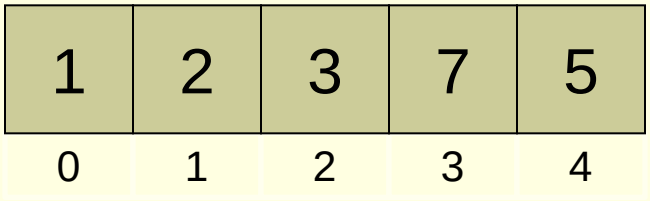
1	2	3	7	5
0	1	2	3	4

i = 4

    - 5 is the value at index 4
      - Compare 5 to the value at index 3 (which is 7)
    - 5 is smaller than 7
      - So we SWAP
    - Again, we are not done

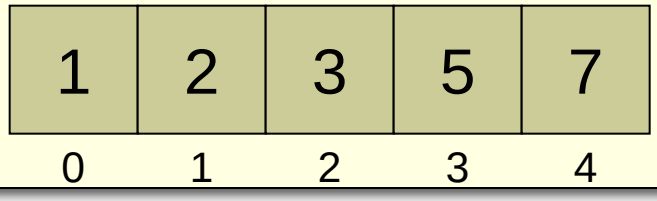
Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Example:
    - Here is an array of 5 integers



i = 4

    - We must now compare 5 to the next element before it
    - So compare 5 to 3
    - 5 is greater than 3, so we can stop and insert 5



Sorting: $O(n^2)$ Algorithms

- **Insertion Sort:**
  - Analysis of Running Time:
    - The number of steps varies based on the input
    - If the list is already in sorted order (best case)
      - During each iteration, the ith element is only compared with one previous element
      - This results in a linear run-time, or O(n)
    - If the list is sorted in reverse order (worst case)
      - During each iteration, the ith element will have to go all the way over to the left
        - During each iteration, the entire, sorted subsection of the array will be shifted over to allow the ith element to go into the front
        - This results in a quadratic run-time, or $O(n^2)$
    - We care about worst case; Insertion Sort runs in $O(n^2)$.

Brief Interlude: FAIL Picture

Sorting: $O(n^2)$ Algorithms

- **Bubble Sort:**
  - Basic idea:
    - You always compare consecutive elements
      - Going left to right
    - Whenever two elements are out of place,
      - SWAP them
    - At the end of a single iteration,
      - the maximum element will be in the last spot
    - Now you simply repeat this n times
      - where n is the number of elements being sorted
    - On each pass, one more maximal element will be put into place

Sorting: $O(n^2)$ Algorithms

- **Bubble Sort:**
  - Example:
    - Here is an array of 8 integers: 6, 2, 5, 7, 3, 8, 4, 1
    - On a single pass of the algorithm, here is the state of the array:

      2, 6, 5, 7, 3, 8, 4, 1
      2, 5, 6, 7, 3, 8, 4, 1
      2, 5, 6, 7, 3, 8, 4, 1
      2, 5, 6, 3, 7, 8, 4, 1
      2, 5, 6, 3, 7, 8, 4, 1
      2, 5, 6, 3, 7, 4, 8, 1
      2, 5, 6, 3, 7, 4, 1, 8 (8 is now in place!)

The "swapped" elements are underlined.

Of course, a swap only occurs as needed.

Note:
We'd have to do this **EIGHT** more times to guarantee a sorted list!

Sorting: $O(n^2)$ Algorithms

- **Bubble Sort:**
  - Truth about Bubble Sort:
  - NOBODY uses Bubble Sort
  - NOBODY.
  - EVER.
  - ‘cept this guy:

  - Reason:
  - cuz Bubble sort is extremely inefficient

Sorting: $O(n^2)$ Algorithms

- **Sorts that only swap adjacent elements**
  - Selection, Insertion, and Bubble sort are examples of sorts where we swap adjacent elements
  - LIMITATION of these types of sorts:
    - They can only run so fast.
  - We can see this once we define an inversion:
    - Inversion: a pair of numbers in a list that is out of order
    - Given this list: 3, 1, 8, 4, 5
    - The inversions are the following pairs of numbers:
      - (3,1), (8, 4), and then (8, 5)

Sorting: $O(n^2)$ Algorithms

- **Sorts that only swap adjacent elements**
  - LIMITATION of these types of sorts:
    - They can only run so fast.
  - We can see this once we define an inversion:
    - When we swap adjacent elements in an array
      - We can remove AT MOST one inversion from the array
    - Now, if it were possible to swap non-adjacent elements,
      - We could remove multiple inversions at the same time
      - Consider the following list: 8, 2, 3, 4, 5, 6, 7, 1
        - Only 8 and 1 are out of order
        - Swapping these two values would remove every inversion
        - It would normally require 13 inversions to get the list sorted if we were limited to swapping only adjacent elements

Sorting: $O(n^2)$ Algorithms

- **Sorts that only swap adjacent elements**
  - Run-time Analysis:
    - Any sorting algorithm that swaps adjacent elements is constrained by the total number of inversions in that array
    - Consider the average case:
      - How many pairs of numbers are there in a list of n numbers?

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

You learn this in Discrete and certain Math courses. For now, just trust me on this.

      - Of these pairs, on average, HALF of them will be inverted.

$$\frac{n(n-1)}{4}$$

We simply divided the previous amount by 2, thus leaving HALF of the pairs left.

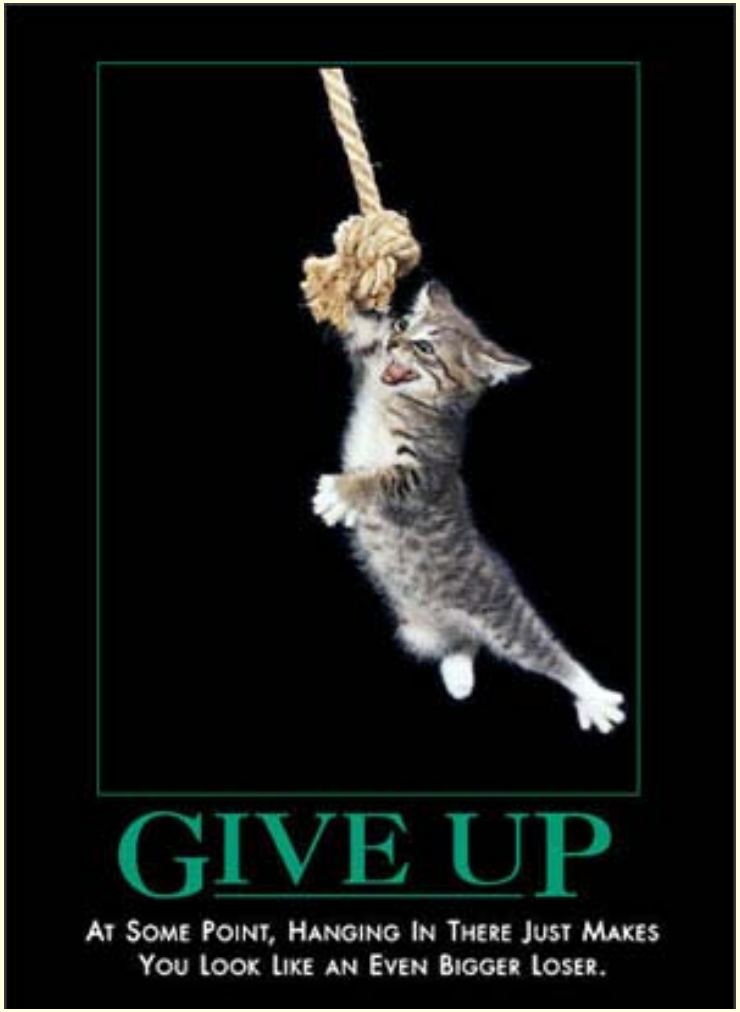
Sorting: $O(n^2)$ Algorithms

- **Sorts that only swap adjacent elements**
  - Run-time Analysis:
    - So, on average, an unsorted array will have $\frac{n(n-1)}{4}$ inversions
    - **Therefore, any sorting algorithm that only swaps adjacent elements will have an $O(n^2)$ run-time.**

Sorting: $O(n^2)$ Algorithms

WASN'T THAT AMAZING!

Daily Demotivator



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