Fun With Summations

Computer Science Department
University of Central Florida

COP 3502 – Computer Science I

Announcement

- Summation Homework
  - There is a NEW, albeit small, homework assignment
    - Just summation problems to work out
  - It is on the course website under “Assignments”
  - Purpose: force you to practice summations
  - Why?
    - Many reasons
    - Including the FACT that they will be on Exam 1!
  - **Due date: IN CLASS, on Wednesday**
    - (meaning in two days, on 2/15)
  - Exam is Friday! Start studying now!

Summations

- Is this a Math class?
- Why do we study summations?
  - In order to effectively approximate algorithms,
  - we NEED mathematical tools
    - It is not always as simply as doing a 4 second examination of a for loop and deciphering the Big-O time
  - <u>So for **iterative** algorithms</u>
    - We use **<u>summations</u>** as the tool (discussed today)
  - For <u>recursive</u> algorithms, this doesn't work
    - We need yet another tool
    - <u>Recurrence relations</u> (coming after the exam)

Summations

- Definition:
  - In very basic terms, a summation is the addition of a set of numbers.
- Example:
  - Let's say we want to sum the integers from 1 to 5
  - Here is how you write this out as a summation:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

  - Here, the "i" is just a variable.
  - Let's look at this notation in more detail…

Summations

- A summation:

So what is f(j)?

upper limit

$$\sum_{j=m}^{n} f(j) \quad \text{or} \quad \sum\nolimits_{j=m}^{n} f(j)$$

lower limit

index of summation

- Here, f(j) is simply a function in terms of j
  - Just like f(x) = 2x+1 is a function in terms of x.
  - f(j) is simply some function in terms of j.

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += f(j);
```

Summations

- Example con'd:
  - Since "i" is just a variable, we can use any variable name…

$$\sum_{Jason=1}^{5} Jason = 1 + 2 + 3 + 4 + 5 = 15$$

  - We also recognize that a summation is merely summing (adding) the values of some given function
  - Thus far, we've only looked at this most simple function:
    - f(i) = i
    - And then we summed up those i terms.

Summations

- Example 2:
  - Now, let us choose our function to be i^2, and let us again sum this from 1 to 5.

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

    - So, the deal here is we are summing whatever the function is, and we are summing this from the lower limit all the way to the upper limit, adding all the values together

  - What if we let the function be 2i+1

$$\sum_{i=1}^{5} 2i+1 = 3+5+7+9+11 = 35$$

Summations

- More summations:
  - Now let us write this purely in "function" form so we all see what is going on.

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5)$$

    - Again, on the previous example, we let f(i) = 2i+1
    - Thus, we had…

$$\sum_{i=1}^{5} 2i+1 = (2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)$$

$$= 3+5+7+9+11 = 35$$

Summations

- Summation Rules:
  - Here are several useful rules of summations:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} k = k\sum_{i=1}^{n} 1 = k * n \qquad \sum_{i=m}^{n} f(i) + g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$$

*k is a constant

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i) \qquad \sum_{i=2}^{5} 2^i$$

Summations

- Examples: $\sum_{i=1}^{n} 42$
  - Here, 42 is a just a constant
  - We can bring it outside of the summation

$$\sum_{i=1}^{n} 42 = 42 \sum_{i=1}^{n} 1$$

  - And now we can solve using this rule: $\sum_{i=1}^{n} 1 = n$

$$\sum_{i=1}^{n} 42 = 42 \sum_{i=1}^{n} 1 = 42n$$

Summations

- Examples: $\sum_{i=1}^{n} 42j$
  - Again, we need to determine, for the "stuff" inside the summation, what are **constants** and what are **variables**.
    - We know **42 is a constant**.
    - What about j?
    - **j is also a constant**!
    - Why?
    - The only time that letters (inside the summation) count as **variables** are when they are the EXACT SAME letter as the index of the summation

Summations

- Examples: $\sum_{i=1}^{n} 42j$
  - Again, we need to determine, <u>for the "stuff" inside the summation</u>, <u>what are **constants**</u> and <u>what are **variables**</u>.
    - i is the index of the summation
    - Do you see any i's inside the summation?
    - Nope.
    - So <u>**ANY**</u> and <u>**ALL**</u> <u>other letters</u> (or anything that looks like it could be a variable) are <u>**NOT variables**</u>
    - Treat them as constants!

Summations

- Examples: $\sum_{i=1}^{n} 42j$
  - In summary, treat the entire 42j as a constant

$$\sum_{i=1}^{n} 42j = 42j\sum_{i=1}^{n} 1$$

  - Now we can solve using the rule: $\sum_{i=1}^{n} 1 = n$

$$\sum_{i=1}^{n} 42j = 42j\sum_{i=1}^{n} 1 = 42j * n$$

Summations

- Examples: $\sum_{Bob=1}^{n} Jeff$
  - Again, look at what is inside the summation
    - All we have is “Jeff” inside the summation
    - The index of the summation is “Bob”
    - So “Jeff” is simply a constant

$$Jeff \sum_{Bob=1}^{n} 1 = Jeff * n$$

Summations

- Additional Problems:
  - We now give several example problems
  - For each problem, we give you the summation RULES that you need for the specific problem
  - These are RULES you will need to know for exams
    - Learn ‘em, memorize ‘em, or include them on your 1 page sheet
  - You then use the rules to solve the problem

Summations

- Problem 1. Evaluate: $\sum_{i=0}^{3}(5+\sqrt{4^i})$
  - The purpose of this example is to simply show that when the limits of the summation are small, you can often ignore rules and just apply an old-school “plug and chug”

Use: $\sum_{i=1}^{n} 1 = n$ $\quad\quad$ $\sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$

$$\sum_{i=0}^{3}(5+\sqrt{4^i}) = (5+\sqrt{4^0}) + (5+\sqrt{4^1}) + (5+\sqrt{4^2}) + (5+\sqrt{4^3})$$

$$= (5+\sqrt{1}) + (5+\sqrt{4}) + (5+\sqrt{16}) + (5+\sqrt{64})$$

= (5+1) + (5+2) + (5+4) + (5+8)

= 6 + 7 + 9 + 13

= 35

Summations

- Problem 2. Evaluate: $\sum_{i=1}^{100}(4+3i)$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i) + g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{100}(4+3i) = \sum_{i=1}^{100} 4 + \sum_{i=1}^{100} 3i$$

$$= \sum_{i=1}^{100} 4 + 3\left(\sum_{i=1}^{100} i\right)$$

$$= 4(100) + 3\left\{\frac{100(100+1)}{2}\right\}$$

= 400 + 15,150

= 15,550

Summations

- Problem 3. Evaluate: $\sum_{i=1}^{200}(i-3)^2$

Use: $\sum_{i=1}^{n} 1 = n$ $\quad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$ $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{200}(i-3)^2 = \sum_{i=1}^{200}(i^2-6i+9)$$

$$\sum_{i=1}^{n} i\wedge 2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \quad = \sum_{i=1}^{200} i^2 - 6\left(\sum_{i=1}^{200} i\right) + \sum_{i=1}^{200} 9$$

$$= \frac{200(200+1)(400+1)}{6} - 6\left\{\frac{200(200+1)}{2}\right\} + 9(200)$$

= 2,686,700 - 120,600 + 1800

= 2,567,900.

Summations

- Problem 4. Evaluate: $\sum_{i=15}^{150}(4i+1)$
  - What is different on this problem?
  - The index of the summation does NOT start at 1!
  - The rules we know and need to solve this problem all assume that the lower index is 1
  - So we need a rule that will change the bounds of the summation
    - A rule that will somehow get as a 1 as the lower limit

Summations

- Problem 4. Evaluate: $\sum_{i=15}^{150}(4i+1)$

Use: $\sum_{i=1}^{n}1=n$ $\quad \sum_{i=m}^{n}f(i)+g(i)=\sum_{i=m}^{n}f(i)+\sum_{i=m}^{n}g(i)$ $\quad \sum_{i=1}^{n}i=\frac{n(n+1)}{2}$

$$\sum_{i=15}^{150}(4i+1)=\sum_{i=15}^{150}4i+\sum_{i=15}^{150}1$$

$$\sum_{i=m}^{n}f(i)=\sum_{i=1}^{n}f(i)-\sum_{i=1}^{m-1}f(i)$$

$$=4\Big(\sum_{i=15}^{150}i\Big)+\sum_{i=15}^{150}1 \qquad =4\Big(\sum_{i=1}^{150}i-\sum_{i=1}^{14}i\Big)+\Big(\sum_{i=1}^{150}1-\sum_{i=1}^{14}1\Big)$$

$$=4\Big(\frac{150(150+1)}{2}-\frac{14(14+1)}{2}\Big)+((1)(150)-(1)(14))$$

= 4(11,325 - 105) + (136)

= 45,016.

Summations

- Problem 5. Evaluate: $$\sum_{i=10}^{80}(i^3+i^2)$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Use: $$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=m}^{n} f(i)+g(i) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i) \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

$$\sum_{i=10}^{80}(i^3+i^2) = \sum_{i=10}^{80} i^3 + \sum_{i=10}^{80} i^2$$

$$= \left(\sum_{i=1}^{80} i^3 - \sum_{i=1}^{9} i^3\right) + \left(\sum_{i=1}^{80} i^2 - \sum_{i=1}^{9} i^2\right)$$

$$= \left(\frac{80^2(80+1)^2}{4} - \frac{9^2(9+1)^2}{4}\right) + \left(\frac{80(80+1)(160+1)}{6} - \frac{9(9+1)(18+1)}{6}\right)$$

= 10,497,600 - 2025 + 173,880 - 285

= 10,669,170.

Brief Interlude: Human Stupidity

Summations

- Problem 6. Evaluate: $\sum_{i=k}^{n} i$
  - Not all summations result in a number for an answer
  - Often, **the answer has many variables in it**
  - The is called the "**closed form**" of the summation
  - Such is the case for this problem
    - And the remaining ones

Summations

- Problem 6. Evaluate: $\sum_{i=k}^{n} i$
  - i is clearly a variable (within the summation)
  - k is a constant, representing the lower limit of the summation
    - gotta fix that!

Use: $\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$

$$\sum_{i=k}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{k-1} i \quad = \frac{n(n+1)}{2} - \frac{(k-1)(k-1+1)}{2}$$

$$= \frac{n(n+1)}{2} - \frac{(k-1)(k)}{2}$$

Summations

- Problem 7. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{m} c$

  Use: $\sum_{i=1}^{n} k = k * n$

  - Start with the right-most summation
    - Apply the rule above

      $$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

  - Now, think about something:
    - How did we know that c was a constant
      - Well, at least in part, cuz I told you!
      - In addition to this fact, you know it is a constant because **variables** are always the **SAME letter** as the **index** of the summation
      - j was the index of the inner summation
      - Thus, any letter would be a constant if it was other than the letter j

Summations

- Problem 7. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{m} c$

  Use: $\sum_{i=1}^{n} k = k * n$

  - Start with the right-most summation
    - Apply the rule above

      $$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

  - Now, look at c*m
    - We know c is a constant, but what about m?
    - Well, the index of the summation is i
    - <u>Any other letter</u>, <u>INSIDE the summation</u> simply acts as a <u>constant</u>
    - **<u>So c*m is also a constant</u>**

Summations

- Problem 7. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{m} c$

Use: $\sum_{i=1}^{n} k = k * n$

- Start with the right-most summation
  - Apply the rule above

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m$$

- So we apply the same rule again:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c = \sum_{i=1}^{n} c * m = (c * m) * n$$

Summations

- Problem 8. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{i} c$

Use: $\sum_{i=1}^{n} k = k * n$

Use: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- Start with the right-most summation
  - Apply the rule above

$$\sum_{i=1}^{n} \sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i$$

- Now, refer back to the previous example
- Ask yourself, **is c*i a constant**?
- Answer: you <u>must look at the INDEX of the summation</u>
  - i is the index, AND i is in the "body" of the summation
  - Therefore, **<u>i is most certainly a variable and should be treated as such!</u>**

Summations

- Problem 8. Evaluate: $\sum_{i=1}^{n} \sum_{j=1}^{i} c$

Use: $\sum_{i=1}^{n} k = k * n$

Use: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  - Start with the right-most summation
    - Apply the rule above

$$\sum_{i=1}^{n} \sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i$$

  - Now, we pull out the constant, and apply the next rule:

$$\sum_{i=1}^{n} \sum_{j=1}^{i} c = \sum_{i=1}^{n} c * i = c \sum_{i=1}^{n} i = c\left[\frac{n(n+1)}{2}\right]$$

Summations

- Problem 9. Evaluate: $\sum_{j=n-10}^{n} 5jn$

$$= \sum_{j=n-10}^{n} 5jn = 5n \sum_{j=n-10}^{n} j$$

$$= 5n\left(\sum_{j=1}^{n} j - \sum_{j=1}^{n-11} j \right) \quad = 5n\left(\frac{n(n+1)}{2} - \frac{(n-11)(n-10)}{2} \right)$$

$$= 5n\left(\frac{n^2 + n - (n^2 - 21n + 110)}{2} \right) = \frac{5n}{2}(n^2 + n - n^2 + 21n - 110)$$

$$= \frac{5n(22n - 110)}{2} = 55n(n-5)$$

Summations

- Problem 10. Evaluate: $\sum_{i=1}^{n}\sum_{j=1}^{i}(2i)$

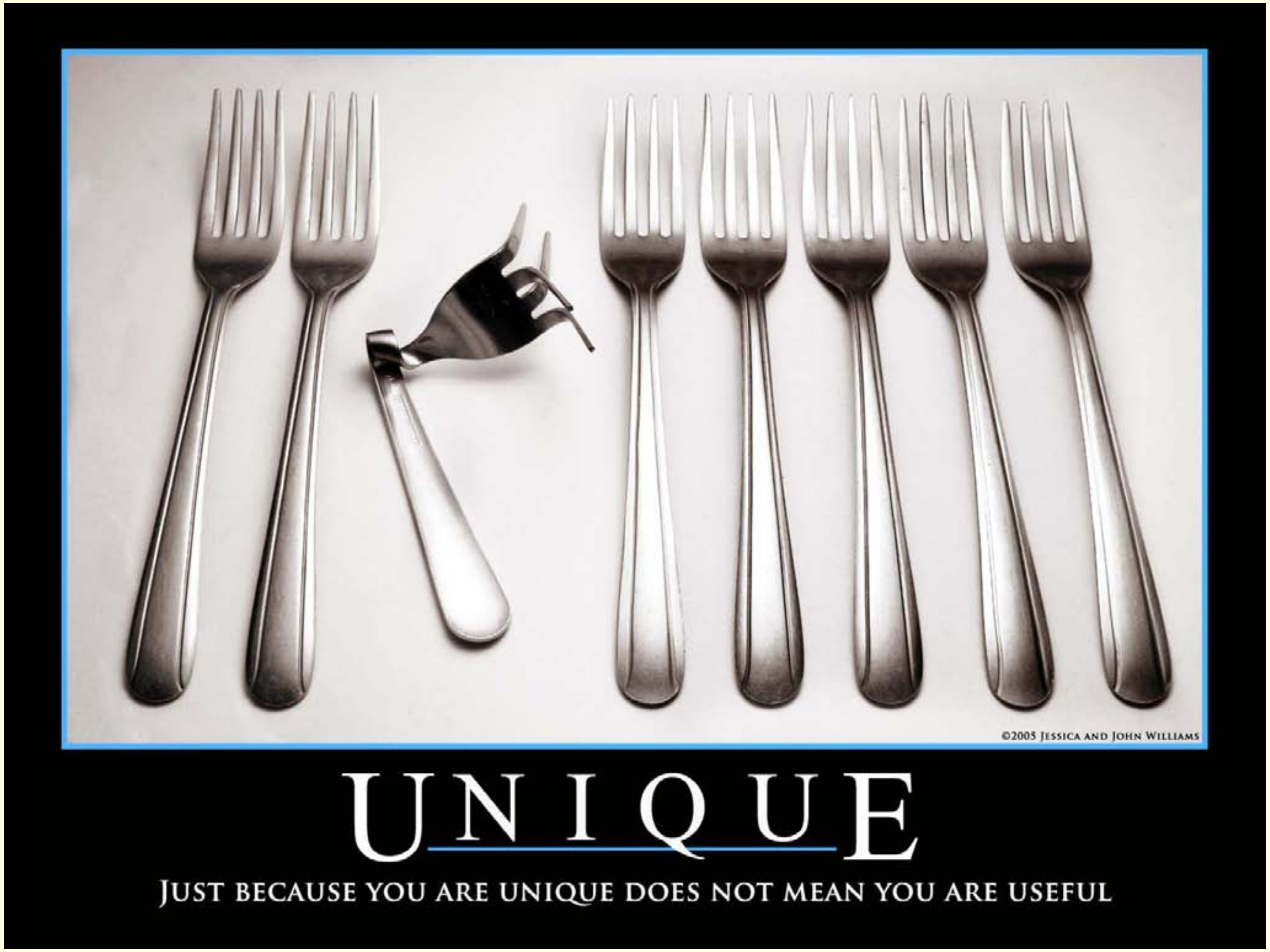
Use: $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$

$$\sum_{i=1}^{n}\sum_{j=1}^{i}(2i) = \sum_{i=1}^{n}2i\sum_{j=1}^{i}1$$

$$= 2\sum_{i=1}^{n}i^2$$

$$= \frac{2n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{3}$$

Daily Demotivator



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