Algorithm Analysis

Computer Science Department
University of Central Florida

COP 3502 – Computer Science I

Order Analysis

- Judging the Efficiency/Speed of an Algorithm
  - Thus far, we've looked at a few different algorithms:
    - Max # of 1's
    - Linear Search vs Binary Search
    - Sorted List Matching Problem
    - and others
  - But we haven't really examined them, in detail, regarding their efficiency or speed
  - **This is one of the main goals of this class!**

Order Analysis

- Judging the Efficiency/Speed of an Algorithm
  - We will use Order Notation to approximate two things about algorithms:
  1) **<u>How much time they take</u>**
  2) How much memory (space) they use
  - Note:
    - It is nearly impossible to figure out the exact amount of time an algorithm will take
    - Each algorithm gets translated into smaller and smaller machine instructions
    - Each of these instructions take various amounts of time to execute on different computers

Order Analysis

- Judging the Efficiency/Speed of an Algorithm
  - Note:
    - Also, <u>we want to judge algorithms independent of their implementation</u>
    - Thus, rather than figure out an algorithm's exact running time
      - <u>**We only want an approximation**</u> (Big-O approximation)
    - Assumptions: we assume that each statement and each comparison in C takes some constant amount of time
    - Also, most algorithms have some type of input
      - With sorting, for example, the size of the input (typically referred to as n) is the number of numbers to be sorted
      - <u>Time and space used by an algorithm function of the input</u>

Big-O Notation

- What is Big O?
  - Sounds like a rapper.?.
    - If it were only that simple!
  - Big O comes from Big-O Notation
    - In C.S., we want to know how efficient an algorithm is…how “fast” it is
    - More specifically…we want to know **how the performance of an algorithm responds to changes in problem size**

Big-O Notation

- What is Big O?
  - The goal is to provide a *qualitative* insight on the # of operations for a problem size of *n* elements.
  - And this total # of operations can be described with a mathematical expression in terms of *n*.
    - This expression is known as Big-O
  - The **Big-O** notation is a way of measuring the order of magnitude of a mathematical expression.
  - O(n) means "of the order of n"

Big-O Notation

- Consider the expression:
  - $f(n) = 4n^2 + 3n + 10$
  - How fast is this "growing"?
    - There are three terms:
      - the $4n^2$, the $3n$, and the 10
    - As n gets bigger, which term makes it get larger fastest?
      - Let's look at some values of n and see what happens?

n	$4n^2$	3n	10
1	4	3	10
10	400	30	10
100	40,000	300	10
1000	4,000,000	3,000	10
10,000	400,000,000	30,000	10
100,000	40,000,000,000	300,000	10
1,000,000	4,000,000,000,000	3,000,000	10

Big-O Notation

- Consider the expression:
  - $f(n) = 4n^2 + 3n + 10$
  - How fast is this "growing"?
    - Which term makes it get larger fastest?
      - As n gets larger and larger, the $4n^2$ term DOMINATES the resulting answer
      - f(1,000,000) = 4,000,003,000,010
  - The idea of behind Big-O is to <u>reduce the expression</u> so that it <u>captures the **qualitative** behavior</u> in the <u>**simplest terms.**</u>

Big-O Notation

- Consider the expression: $f(n) = 4n^2 + 3n + 10$
  - How fast is this "growing"?
    - Look at VERY large values of n
      - **eliminate** any term whose contribution to the total ceases to be significant as n get larger and larger
      - of course, this also includes constants, as they little to no effect with larger values of n
        - Including constant factors (coefficients)
      - So we ignore the constant 10
      - And we can also ignore the 3n
      - Finally, we can eliminate the constant factor, 4, in front of n^2
    - We can approximate the order of this function, f(n), **as n^2**
    - We can say, **$O(4n^2 + 3n + 10) = O(n^2)$**
      - In conclusion, we say that f(n) takes $O(n^2)$ steps to execute

Big-O Notation

- Some basic examples:
  - What is the Big-O of the following functions:
    - $f(n) = 4n^2 + 3n + 10$
      - Answer: $O(n^2)$
    - $f(n) = 76{,}756{,}234n^2 + 427{,}913n + 7$
      - Answer: $O(n^2)$
    - $f(n) = 74n^8 - 62n^5 - 71562n^3 + 3n^2 - 5$
      - Answer: $O(n^8)$
    - $f(n) = 42n^4 * (12n^6 - 73n^2 + 11)$
      - Answer: $O(n^{10})$
    - $f(n) = 75n * \log n - 415$
      - Answer: $O(n * \log n)$

Big-O Notation

- Consider the expression: $f(n) = 4n^2 + 3n + 10$
  - How fast is this "growing"?
    - We can say, $O(4n^2 + 3n + 10) = O(n^2)$
    - Till now, we have one function:
      - $f(n) = 4n^2 + 3n + 10$
    - Let us make a second function, **g(n)**
      - It's just a letter right? We could have called it r(n) or x(n)
        - Don't get scared about this
    - Now, let g(n) equal n^2
      - **$g(n) = n^2$**
    - So now we have two functions: **f(n)** and **g(n)**
      - We said (above) that $O(4n^2 + 3n + 10) = O(n^2)$
    - Similarly, we can say that the **order of f(n) is O[g(n)].**

Big-O Notation

Brace yourself!

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - Think about the two functions we just had:
      - **f(n) = $4n^2$ + 3n + 10**, and **g(n) = n^2**
      - We agreed that O($4n^2$ + 3n + 10) = O(n^2)
      - Which means we agreed that the order of **<u>f(n) is O(g(n)</u>**
    - <u>That's all this definition says!!!</u>
    - f(n) is big-O of g(n), if there is a c,
      - (c is a constant)
    - such that f(n) is not larger than c*g(n) for sufficiently large values of n (greater than N)

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - Think about the two functions we just had:
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - f is big-O of g, if there is a c such that f is not larger than c*g for sufficiently large values of n (greater than N)
      - So given the two functions above, **<u>does there exist</u>** some **<u>constant</u>**, **<u>c</u>**, that would make the following statement true?
      - f(n) <= c*g(n)
      - **$4n^2 + 3n + 10 <= c*n^2$**
      - <u>If there does exist this c</u>, **<u>then f(n) is O(g(n))</u>**
    - Let's go see if we can come up with the constant, c

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **<u>Find the c</u>** such that **$4n^2 + 3n + 10 <= c*n^2$**
    - Clearly, c cannot be 4 or less
      - Cause <u>even if it was 4</u>, we would have:
        - **$4n^2 + 3n + 10 <= 4n^2$**
        - This is <u>NEVER true for any positive value of n</u>!
      - So **<u>c must be greater than 4</u>**
    - Let us <u>try with c being equal to 5</u>
      - **$4n^2 + 3n + 10 <= 5n^2$**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **<u>Find the c</u>** such that **$4n^2 + 3n + 10 <= c*n^2$**
      - **$4n^2 + 3n + 10 <= 5n^2$**
      - For what values of n, if ANY at all, is this true?

n	**$4n^2 + 3n + 10$**	**$5n^2$**
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **<u>Find the c</u>** such that **$4n^2 + 3n + 10 <= c*n^2$**
      - **$4n^2 + 3n + 10 <= 5n^2$**
      - For what values of n, if ANY at all, is this true?

n	**$4n^2 + 3n + 10$**	**$5n^2$**
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**
2	4(4) + 3(2) + 10 = **32**	5(4) = **20**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] if there exists positive integers c and N, such that f(n) <= c*g(n) for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **Find the c** such that **$4n^2 + 3n + 10 \le c*n^2$**
      - **$4n^2 + 3n + 10 \le 5n^2$**
      - For what values of n, if ANY at all, is this true?

n	**$4n^2 + 3n + 10$**	**$5n^2$**
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**
2	4(4) + 3(2) + 10 = **32**	5(4) = **20**
3	4(9) + 3(3) + 10 = **55**	5(9) = **45**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **f(n) = 4n^2 + 3n + 10**, and **g(n) = n^2**
    - **<u>Find the c</u>** such that **4n^2 + 3n + 10 <= c*n^2**
      - **4n^2 + 3n + 10 <= 5n^2**
      - For what values of n, if ANY at all, is this true?

n	4n^2 + 3n + 10	5n^2
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**
2	4(4) + 3(2) + 10 = **32**	5(4) = **20**
3	4(9) + 3(3) + 10 = **55**	5(9) = **45**
4	4(16) + 3(4) + 10 = **86**	5(16) = **80**

But now let's try larger values of n.

- For n = 1 through 4, this statement is NOT true

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **<u>Find the c</u>** such that **$4n^2 + 3n + 10 <= c*n^2$**
      - **$4n^2 + 3n + 10 <= 5n^2$**
      - For what values of n, if ANY at all, is this true?

n	**$4n^2 + 3n + 10$**	**$5n^2$**
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**
2	4(4) + 3(2) + 10 = **32**	5(4) = **20**
3	4(9) + 3(3) + 10 = **55**	5(9) = **45**
4	4(16) + 3(4) + 10 = **86**	5(16) = **80**
5	4(25) + 3(5) + 10 = **125**	5(25) = **125**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **$f(n) = 4n^2 + 3n + 10$**, and **$g(n) = n^2$**
    - **<u>Find the c</u>** such that **$4n^2 + 3n + 10 <= c*n^2$**
      - **$4n^2 + 3n + 10 <= 5n^2$**
      - For what values of n, if ANY at all, is this true?

n	**$4n^2 + 3n + 10$**	**$5n^2$**
1	4(1) + 3(1) + 10 = **17**	5(1) = **5**
2	4(4) + 3(2) + 10 = **32**	5(4) = **20**
3	4(9) + 3(3) + 10 = **55**	5(9) = **45**
4	4(16) + 3(4) + 10 = **86**	5(16) = **80**
5	4(25) + 3(5) + 10 = **125**	5(25) = **125**
6	4(36) + 3(6) + 10 = **172**	5(36) = **180**

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **f(n) = 4n^2 + 3n + 10**, and **g(n) = n^2**
    - **<u>Find the c</u>** such that **4n^2 + 3n + 10 <= c*n^2**
      - **4n^2 + 3n + 10 <= 5n^2**
      - For what values of n, if ANY at all, is this true?
      - So <u>when n = 5</u>, the <u>statement finally becomes **true**</u>
      - And **<u>when n > 5, it remains true</u>**!

    - So our constant, 5, works for all n >= 5.

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - PROBLEM: Given our two functions,
      - **f(n) = 4n^2 + 3n + 10**, and **g(n) = n^2**
    - **<u>Find the c</u>** such that **4n^2 + 3n + 10 <= c*n^2**
    - So our constant, 5, works for all n >= 5.
    - Therefore, **<u>f(n) is O(g(n))</u>** per our definition!
    - Why?
    - Because <u>there exists positive integers, c and N</u>,
      - Just so happens in this case that c = 5 and N = 5
    - **<u>such that f(n) <= c*g(n)</u>**.

Who actually got that ?

Big-O Notation

- Definition:
  - ***f(n) is O[g(n)] <u>if there exists</u> positive integers c and N, such that <u>f(n) <= c*g(n)</u> for all n>=N.***
    - What can we take from this?
      - That Big-O is hard as #$%q@$^&!!!
    - No, but seriously…
    - What we can gather is that:
    - **c*g(n)** is an **<u>upper bound</u>** on the value of **f(n)**.
      - It represents the <u>**worst possible scenario** of running time</u>.
    - The number of operations is, at worst, proportional to g(n) for all <u>large values</u> of n.

Big-O Notation

- Summing up the basic properties for determining the order of a function:
  1) If you’ve got multiple functions added together, the fastest growing one determines the order
  2) Multiplicative constants don’t affect the order
  3) If you’ve got multiple functions multiplied together, the overall order is their individual orders multiplied together

Big-O Notation

- Some basic examples:
  - What is the Big-O of the following functions:
    - $f(n) = 4n^2 + 3n + 10$
      - Answer: $O(n^2)$
    - $f(n) = 76{,}756{,}234n^2 + 427{,}913n + 7$
      - Answer: $O(n^2)$
    - $f(n) = 74n^8 - 62n^5 - 71562n^3 + 3n^2 – 5$
      - Answer: $O(n^8)$
    - $f(n) = 42n^4*(12n^6 - 73n^2 + 11)$
      - Answer: $O(n^{10})$
    - $f(n) = 75n*\log n – 415$
      - Answer: $O(n*\log n)$

Big-O Notation

- Quick Example of Analyzing Code:
  - This is just to show you how we use Big-O
    - we‘ll do more of these (a lot more) next time
  - Use big-O notation to analyze the time complexity of the following fragment of C code:

```
for (k=1; k<=n/2; k++) {
      sum = sum + 5;
}

for (j = 1; j <= n*n; j++) {
      delta = delta + 1;
}
```

Big-O Notation

- Quick Example of Analyzing Code:
  - So look at what's going on in the code:
    - We care about the total number of REPETITIVE operations.
      - Remember, we said we care about the running time for LARGE values of n
      - So in a **for loop**, with n as part of the comparison value determining when to stop

```
for (k=1; k<=n/2; k++)
```

      - Whatever is INSIDE that loop will be executed a LOT of times
      - So we examine the code within this loop and see how many operations we find
        - When we say operations, we're referring to mathematical operations such as +, -, *, /, etc.

Big-O Notation

- Quick Example of Analyzing Code:
  - So look at what’s going on in the code:
    - The number of operations executed by these loops is the sum of the individual loop operations.
    - We have 2 loops,

```
for (k=1; k<=n/2; k++) {
      sum = sum + 5;
}

for (j = 1; j <= n*n; j++) {
      delta = delta + 1;
}
```

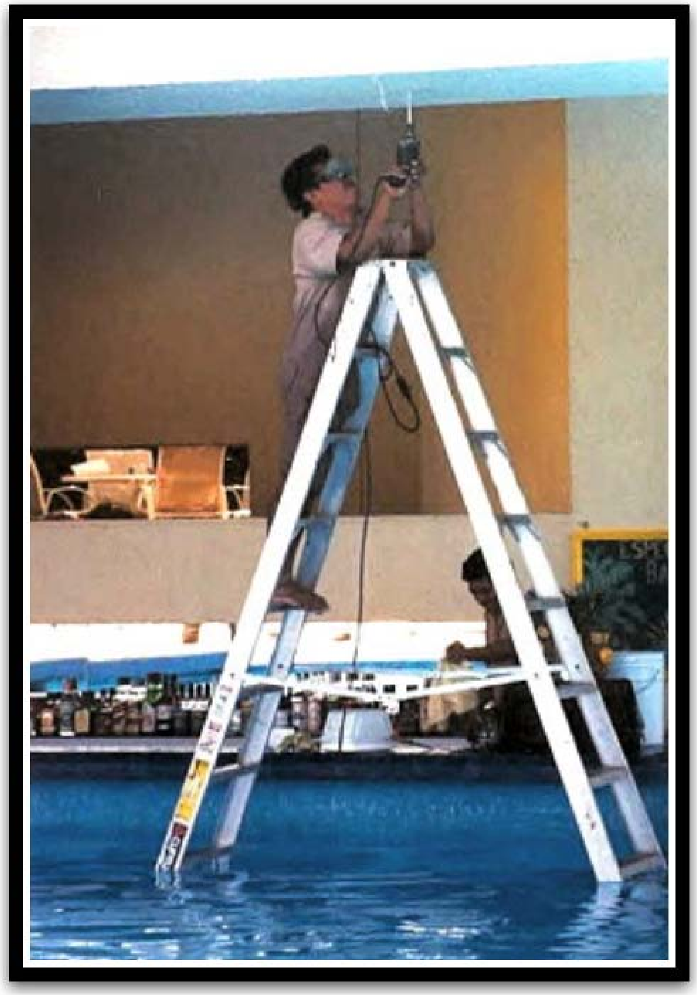
Big-O Notation

- Quick Example of Analyzing Code:
  - So look at what's going on in the code:
    - The number of operations executed by these loops is the sum of the individual loop operations.
    - We have 2 loops,
      - The first loop runs n/2 times
      - Each iteration of the <u>first loop</u> results in <u>one operation</u>
        - The + operation in: `sum = sum + 5;`
      - So there are n/2 operations in the first loop
      - The second loop runs n^2 times
      - Each iteration of the <u>second loop</u> results in <u>one operation</u>
        - The + operation in: `delta = delta + 1;`
      - So there are n^2 operations in the second loop.

Big-O Notation

- Quick Example of Analyzing Code:
  - So look at what's going on in the code:
    - The number of operations executed by these loops is the sum of the individual loop operations.
    - The first loop has n/2 operations
    - The second loop has n^2 operations
    - They are NOT nested loops.
      - One loop executes AFTER the other completely finishes
    - So <u>we simply ADD their operations</u>
    - The total number of operations would be $n/2 + n^2$
    - In Big-O terms, we can express the number of operations as $O(n^2)$

Brief Interlude: Human Stupidity



Big-O Notation

- Common orders (listed from slowest to fastest growth)

Function	Name
1	Constant
log n	Logarithmic
n	Linear
n log n	Poly-log
n^2	Quadratic
n^3	Cubic
2^n	Exponential
n!	Factorial

Big-O Notation

- **<u>O(1)</u>** or “Order One”: **<u>Constant time</u>**
  - <u>does not mean</u> that it takes <u>only one operation</u>
  - does mean that the work doesn’t change as n changes
  - is a notation for “<u>constant work</u>”
  - An example would be finding the smallest element in a sorted array
    - There’s nothing to search for here
    - The smallest element is always at the beginning of a sorted array
    - So this would take O(1) time

Big-O Notation

- **O(n)** or "Order n": **Linear time**
  - does not mean that it takes n operations
    - maybe it takes 3*n operations, or perhaps 7*n operations
  - does mean that the **work changes in a way that is proportional to n**
  - Example:
    - If the input size doubles, the running time also doubles
  - is a notation for "**work grows at a linear rate**"
  - You usually can't really do a lot better than this for most problems we deal with
    - After all, you need to at least examine all the data right?

Big-O Notation

- **$O(n^2)$** or "Order n^2 ": **Quadratic time**
  - If input size doubles, running time increases by a factor of 4
- **$O(n^3)$** or "Order n^3 ": **Cubic time**
  - If input size doubles, running time increases by a factor of 8
- **$O(n^k)$**: Other polynomial time
  - Should really try to avoid high order polynomial running times
    - However, it is considered good from a theoretical standpoint

Big-O Notation

- **<u>O(2^n)</u>** or "Order 2^n ": **<u>Exponential time</u>**
  - more <u>theoretical</u> rather than practical interest because they cannot reasonably run on typical computers for even for moderate values of n.
  - Input sizes bigger than 40 or 50 become <u>unmanageable</u>
    - Even on faster computers
- **<u>O(n!)</u>**: even worse than exponential!
  - Input sizes bigger than 10 will take a long time

Big-O Notation

- **O(n logn)**:
  - Only slightly worse than O(n) time
    - And O(n logn) will be much less than O(n^2)
    - This is the running time for the better sorting algorithms we will go over (later)
- **O(log n)** or “Order log n”: **Logarithmic time**
  - If input size doubles, running time increases ONLY by a constant amount
  - **any algorithm that halves the data** remaining to be processed on each iteration of a loop will be an **O(log n) algorithm**.

Big-O Notation – Practical Problems

- Practical Problems that can be solved utilizing order notation:
  - Example:
    - You are told that algorithm A runs in <u>O(n) time</u>
    - You are also told the following:
      - For an <u>input size of 10</u>
      - The algorithm runs in <u>2 milliseconds</u>
    - As a result, you can expect that for an input size of 500, the algorithm would run in 100 milliseconds!
      - Notice the <u>input size jumped by a multiple of 50</u>
        - From 10 to 500
      - Therefore, given a O(n) algorithm, the <u>running time should also jump by a multiple of 50</u>, <u>**which it does**</u>!

Big-O Notation – Practical Problems

- Practical Problems that can be solved utilizing order notation:
  - General process of solving these problems:
    - We know that Big-O is NOT exact
      - It's an upper bound on the actual running time
    - So when we say that an **algorithm runs in O(f(n)) time**,
    - **Assume** the EXACT **running time is c*f(n)**
      - where c is some constant
    - Using this assumption,
      - we can use the information in the problem to solve for c
      - Then we can use this c to answer the question being asked
    - Examples will clarify…

Big-O Notation – Practical Problems

- Practical Problems that can be solved utilizing order notation:
  - Example 1: Algorithm A runs in **$O(n^2)$** time
    - For an input size of 4, the running time is 10 milliseconds
    - How long will it take to run on an input size of 16?
    - **Let $T(n) = c*n^2$**
      - T(n) refers to the running time (of algorithm A) on input size **n**
      - Now, plug in the given data, and **find the value for c!**
    - $T(4) = c*4^2 = 10$ milliseconds
      - Therefore, **c = 10/16 milliseconds**
    - Now, answer the question by **using c** and **solving T(16)**
    - **$T(16) = c*16^2$** $= (10/16)*16^2 = 160$ milliseconds

Big-O Notation – Practical Problems

- Practical Problems that can be solved utilizing order notation:
  - Example 2: Algorithm A runs in **$O(\log_2 n)$** time
    - For an input size of 16, the running time is 28 milliseconds
    - How long will it take to run on an input size of 64?
    - **Let $T(n) = c*\log_2 n$**
      - Now, plug in the given data, and **find the value for c!**
    - $T(16) = c*\log_2 16 = 10$ milliseconds
      - $c*4 = 28$ milliseconds
      - Therefore, **c = 7 milliseconds**
    - Now, answer the question by **using c** and **solving T(64)**
    - **$T(64) = c*\log_2 64$** $= 7*\log_2 64 = 7*6 = 42$ milliseconds

Base Conversions

WASN'T THAT MARVELOUS!

Daily Demotivator

CHANGE
It's a short trip from riding the waves of change to being torn apart by the jaws of defeat.

Algorithm Analysis

Computer Science Department
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COP 3502 – Computer Science I