**Computer Science I – Spring 2012**
**Lab: Recurrence Relations (*Solutions*)**

Solve the following recurrence relations using the iteration technique:

**1)** $T(n) = T(n-1) + 7, \quad T(1) = 4$

==================================================

T(n) = T(n-1) + 7
T(1) = 4

T(n) = T(n-1) + 7 = [T(n-2) + 7] + 7 = T(n-2) + 7 + 7
T(n) = T(n-2) + 2*7
T(n) = T(n-2) + 2*7 = [T(n-3) + 7] + 2*7 = T(n-3) + 7 + 2*7
T(n) = T(n-3) + 3*7
T(n) = T(n-3) + 3*7 = [T(n-4) + 7] + 3*7 = T(n-4) + 7 + 3*7
T(n) = T(n-4) + 4*7

Substituting Equations
$n \rightarrow n-1$

T(n-1) = T(n-2) + 7
T(n-2) = T(n-3) + 7
T(n-3) = T(n-4) + 7
T(n-4) = T(n-5) + 7

Do it one more time…

T(n) = T(n-5) + 5*7

So now rewrite these five equations and look for a pattern:

| | | |
|---|---|---|
| T(n) = T(n-1) + 1*7 | ⟵ | $1^{st}$ step of recursion |
| T(n) = T(n-2) + 2*7 | ⟵ | $2^{nd}$ step of recursion |
| T(n) = T(n-3) + 3*7 | ⟵ | $3^{rd}$ step of recursion |
| T(n) = T(n-4) + 4*7 | ⟵ | $4^{th}$ step of recursion |
| T(n) = T(n-5) + 5*7 | ⟵ | $5^{th}$ step of recursion |

Generalized recurrence relation at the kth step of the recursion:
T(n) = T(n-k) + k*7

We want T(1). So we let n-k = 1. Solving for k, we get k = n - 1. Now plug back in.

T(n) = T(n-k) + k*7
T(n) = T(1) + 7*(n-1), and we know T(1) = 4
T(n) = 4 + 7*(n-1) = 7n - 7 + 4 = 7n - 3

We are done. Right side does not have any T(…)'s. This recurrence relation is now solved in its closed form, and it runs in O(n) time.

2) $T(n) = 2T\left(\frac{n}{2}\right) + 2, T(1) = 2$

=================================================

**T(n) = 2T(n/2) + 2**
T(1) = 2

Substituting Equations
n → n/2

T(n/2) = 2T(n/4) + 2
T(n/4) = 2T(n/8) + 2
T(n/8) = 2T(n/16) + 2
T(n/16) = 2T(n/32) + 2

T(n) = 2T(n/2) + 2 = 2[2T(n/4) + 2] + 2 = 4T(n/4) + 4 + 2
**T(n) = 4T(n/4) + 6**

T(n) = 4T(n/4) + 6 = 4[2T(n/8) + 2] + 6 = 8T(n/8) + 8 + 6
**T(n) = 8T(n/8) + 14**

T(n) = 8T(n/8) + 14 = 8[2T(n/16) + 2] + 14 = 16T(n/16) + 16 + 14
**T(n) = 16T(n/16) + 30**

T(n) = 16T(n/16) + 30 = 16[2T(n/32) + 2] + 30 = 32T(n/32) + 32 + 30
**T(n) = 32T(n/32) + 62**

Do it again:
**T(n) = 64T(n/64) + 126**

So now rewrite these five equations and look for a pattern:

| | | | |
|---|---|---|---|
| T(n) = 2T(n/2) + 2 | = $\mathbf{2^1T(n/2^1) + 2^2-2}$ | ← | 1st step of recursion |
| T(n) = 4T(n/4) + 6 | = $\mathbf{2^2T(n/2^2) + 2^3-2}$ | ← | 2nd step of recursion |
| T(n) = 8T(n/8) + 14 | = $\mathbf{2^3T(n/2^3) + 2^4-2}$ | ← | 3rd step of recursion |
| T(n) = 16T(n/16) + 30 | = $\mathbf{2^4T(n/2^4) + 2^5-2}$ | ← | 4th step of recursion |
| T(n) = 32T(n/32) + 62 | = $\mathbf{2^5T(n/2^5) + 2^6-2}$ | ← | 5th step of recursion |
| **T**(n) = 64T(n/64) + 126 | = $\mathbf{2^6T(n/2^6) + 2^7-2}$ | ← | 6th step of recursion |

**In general, after k iterations, we have:**

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^{k+1} - 2$$

We're not done since we still have T(…)'s on the right side of the equation. We need to get down to T(1). How?

We have $T(n/2^k)$, and we want T(1). So let $n = 2^k$. We will then have $T(2^k/2^k)$, which equals T(1). So use that substitution ($n = 2^k$) throughout the entire generalized, kth recurrence relation.

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^{k+1} - 1 = n * T\left(\frac{2^k}{2^k}\right) + 2n - 1 = n * T(1) + 2n - 1$$

$$T(n) = n * 2 + 2n - 1 = 4n - 1$$

So, T(n) = 4n – 1 and runs in O(n) time.

3) $T(n) = T\left(\frac{n}{2}\right) + n, T(1) = 1$, Hint: $\sum_{i=0}^{\infty} \frac{n}{2^i} = 2n$ (Just get an approximate solution here.)

==================================================

$$T(n) = T\left(\frac{n}{2}\right) + n$$
$$T(n) = T\left(\frac{n}{4}\right) + \frac{n}{2} + n$$
$$T(n) = T\left(\frac{n}{8}\right) + \frac{n}{4} + \frac{n}{2} + n$$

Here's the generalized recurrence relation for the k[th] step of the recursion:

$$T(n) = T\left(\frac{n}{2^k}\right) + \boldsymbol{\frac{n}{2^{k-1}} + \frac{n}{2^{k-2}} + \cdots \frac{n}{2^3} + \frac{n}{2^2} + \frac{n}{2^1} + n}$$

Look at the bolded portion: let's refer to it as the "stuff remaining". Unfortunately, there is no direct way to represent this series. Like on previous, simple examples, if we had "k" 7's on step k of the recursion, we would simply represent that as k*7. But, it is not that easy here.

So what do you do? **Express (write out) that series as a summation**, where the step of the recursion is part of the limit of the summation. So on the 3[rd] step of the recursion, we had these three expressions that were added together:

$$\frac{n}{4} + \frac{n}{2} + n$$

Remember, this is the 3[rd] step of the recursion. How can you write this as a summation?

$$\sum_{i=0}^{2} \frac{n}{2^i} = \frac{n}{2^0} + \frac{n}{2^1} + \frac{n}{2^2} = \frac{n}{1} + \frac{n}{2} + \frac{n}{4} = n + \frac{n}{2} + \frac{n}{4}$$

So now that we've represented the "stuff" for the 3[rd] step of recursion. Try it for the 117[th] step!

$$\sum_{i=0}^{116} \frac{n}{2^i}$$

Finally, express this "remaining stuff" for the k[th] step:

$$\sum_{i=0}^{k-1} \frac{n}{2^i}$$

So, at the kth step of the recursion, we have the following generalized recurrence relation:

$$T(n) = T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} \frac{n}{2^i}$$

Now, we can solve; but we will need to make use of the hint given at the beginning!

Using the hint, we find that this summation is approximately 2n. (It's more accurately 2n – 2, but this is a very minor difference from the infinite sum given.)

$$T(n) \sim T\left(\frac{n}{2^k}\right) + 2n$$

We have $T(n/2^k)$, and we want T(1). So like on the last example, let $n = 2^k$. We will then have $T(2^k/2^k)$, which equals T(1). Plug in:

$$T(n) \sim T\left(\frac{2^k}{2^k}\right) + 2n = T(1) + 2n$$

$$T(n) \sim 1 + 2n$$

So this also runs in O(n) time.