COP 3502: Computer Science I
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– Day 9 –
Algorithm Analysis – Part 3

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Dealing with Summations

- Most programs contain some sort of looping constructs which causes a repetition of the execution of some program instructions.
- When analyzing running time costs for programs with loops, we need to add up the costs for each time the loop is executed.
- This is an example of a summation. Summations are simply the sum for some function over a range of parameter values.
- Summations are typically written using the "Sigma" notation:

$$\boxed{\sum_{i=1}^{n} f(i)}$$

Dealing with Summations (cont.)

$$\sum_{i=1}^{n} f(i)$$

- The Sigma notation indicates that we are summing the value of *f(i)* over some range of integer values.
- The parameter to the expression and its initial value are indicated below the Σ symbol.
  - In this case the notation *i = 1* indicates that the parameter (summation variable) is *i* and its initial value is *1*.
- On top of the Σ symbol is the expression which indicates the maximum value for the summation variable.
  - In this case, this expression is simply *n*.

Dealing with Summations (cont.)

- Note that in many textbooks that the Sigma notation is often typeset inline as: $\sum_{i=1}^{n} f(i)$

$$\boxed{\sum_{i=1}^{n} f(i) \equiv f(1) + f(2) + \cdots + f(n-1) + f(n)}$$

- Given a summation, it is often replaced with an equation that directly computes the value of the summation.

- Such an equation is known as a closed-form solution, and the process of replacing the summation with its closed-form solution is known as solving the summation.

Common Summations and Their Closed-Form Solution

- Below are the some very common summations and their closed-form.

(1) $$\sum_{i=1}^{n} 1 = n$$

(2) $$\sum_{i=0}^{n} 1 = n + 1$$

(3) $$\sum_{i=0}^{n} i \equiv \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

(4) $$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

(5) $$\sum_{i=1}^{\log n} n = n \log n$$

(6) $$\sum_{i=1}^{n} \frac{i}{2^i} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

Generating Closed-Form Solutions for Summations

- The question is, how do you generate the closed-form solution for a summation?
- The answer is: by solving the summation.
- The technique used to solve the summation varies slightly depending on the summation. There are a few general cases to consider:
  - summations which begin at either 0 or 1 and have a constant upper limit.
  - summations which begin at either 0 or 1 and have a variable upper limit. If the upper summation limit is a variable, then the summation can be expressed in a closed-form.
  - summations which begin at a value other than 0 or 1 and have either a constant or variable upper limit.
- We'll look at all these cases separately.

Constant Upper Limit to the Summation

- *Examples:* Given the following summations, evaluate the summations.

$$\sum_{i=0}^{6} 1 = \sum_{i=0}^{6} 1 = (1+1+1+1+1+1+1) = 7$$

$$\sum_{i=1}^{6} 1 = \sum_{i=1}^{6} 1 = (1+1+1+1+1+1) = 6$$

Note: we could have evaluated the summations using the closed-form solution of the summation.

$$\sum_{i=0}^{n} 1 = (n+1) = (6+1) = 7$$

$$\sum_{i=1}^{n} i = n = 6$$

Constant Upper Limit to the Summation

- *Example:* Given the following summation, evaluate the summation.

$$\sum_{i=1}^{6} 2i = 2\sum_{i=1}^{6} i = 2\times(1+2+3+4+5+6) = 2\times 21 = 42$$

constants are not part of the summation and can be removed from it.

Note: we could have evaluated the summation using the closed-form solution of the summation.

$$\sum_{i=0}^{n} 2i = 2\sum_{i=0}^{n} i = \frac{2(n(n+1))}{2} = \frac{2(6^2)+2(6)}{2} = 36+6 = 42$$

Variable Upper Limit to the Summation

- *Example:* Given the following summation, produce the closed-form solution.

$$\sum_{i=1}^{n} 5i = 5\sum_{i=1}^{n} i = \frac{5(n(n+1))}{2} = \frac{5n^2 + 5n}{2}$$

Note: the following summation produces the same result!

$$\sum_{i=0}^{n} 5i = 5\sum_{i=0}^{n} i = \frac{5(n(n+1))}{2} = \frac{5n^2 + 5n}{2}$$

The first term in this summation adds nothing to the result since $5 \times 0 = 0$

Once the closed form is found, we can evaluate the summation for some given upper limit on the summation. In this example, if we assume that n = 5, then we have:

$$\sum_{i=1}^{n} 5i = \frac{5n^2 + 5n}{2} = \frac{5(25) + 25}{2} = \frac{150}{2} = 75$$

Generating Closed-Forms for Complex Functions

- If the summation function involves additional terms, then the summation is broken-up into separate summations as shown in the example below:

- *Examples:*

$$\sum_{i=1}^{n}(4i+8) = 4\sum_{i=1}^{n} i + 8\sum_{i=1}^{n} 1 = \frac{4(n(n+1))}{2} + 8n = \frac{4n^2+4n}{2} + 8n = 2n^2 + 10n$$

initial function (4i+8) is split into two pieces.

$$\sum_{i=0}^{n}(4i+8) = 4\sum_{i=0}^{n} i + 8\sum_{i=0}^{n} 1 = \frac{4(n(n+1))}{2} + 8(n+1) = \frac{4n^2+4n}{2} + 8n + 8 = 2n^2 + 10n + 8$$

Generating Closed-Forms for Complex Functions

- *A couple more examples:*

$$\sum_{i=1}^{n}(3i-5)=3\sum_{i=1}^{n}i-5\sum_{i=1}^{n}1=\frac{3(n(n+1))}{2}-5n=\frac{3n^2+3n}{2}-\frac{10n}{2}=\frac{3n^2-7n}{2}$$

$$\sum_{i=0}^{n}\left(4i^2+3i-2\right)=4\sum_{i=0}^{n}i^2+3\sum_{i=0}^{n}i-2\sum_{i=0}^{n}1=$$

$$=\frac{4\left(2n^3+3n^2+n\right)}{6}+\frac{3\left(n^2+n\right)}{2}-2(n+1)=\frac{8n^3+12n^2+4n}{6}+\frac{3n^2+3n}{2}-2n-2=$$

$$=\frac{8n^3+12n^2+4n+9n^2+9n-12n-12}{6}=\frac{8n^3+21n^2+n-12}{6}$$

Closed-form for Summations with Initial Value > 1

- There are two basic techniques that can be used to determine the closed-form solution for summations in which the initial value of the summation is greater than 1.

$$\sum_{i=m}^{n} f(i)$$

1. Shift the range of the summation.
   - In this technique you shift the summation range down until the starting value is equal to 1 and add into the function all of the values that are omitted because of the range shift.
2. Subtract the difference between two summations.
   - In this technique you actually produce the summation over the entire range from 1 to n and then subtract out the value of the summation over the range from 1 to m-1.

- We'll look at some examples using both techniques and then you can decide for yourself which technique you prefer.

Shifting the Range of the Summation

- The following example illustrates the range shifting technique for dealing with summations where the initial value of the summation range is greater than 1.
- *Example:*
- Notice that there are 7 terms in: $\sum_{i=4}^{10} i = (4+5+6+7+8+9+10)$
- the difference between those seven numbers and the summation

  $\sum_{i=1}^{7} i = (1+2+3+4+5+6+7)$

  is that the numbers in (4+5+6+7+8+9+10) are each 3 larger than the numbers in (1+2+3+4+5+6+7) .
- Thus, (4+5+6+7+8+9+10) is equal to

  (1+3)+(2+3)+(3+3)+(4+3)+(5+3)+(6+3)+(7+3)
- We can write (1+3)+(2+3)+(3+3)+(4+3)+(5+3)+(6+3)+(7+3) as

Shifting the Range of the Summation

$$\sum_{i=1}^{7}(i+3) = ((1+3)+(2+3)+(3+3)+(4+3)+(5+3)+(6+3)+(7+3))$$

This means that:

$$\sum_{i=4}^{10} i \;=\; \sum_{i=1}^{7}(i+3) = 49$$

$$\sum_{i=1}^{7}(i+3) = \sum_{i=1}^{7} i + 3\sum_{i=1}^{7} 1 = \frac{(n^2+n)}{2} + 3(n) = \frac{49+7}{2} + 21 = \frac{56}{2} + 21 = 49$$

Let's check this to be certain:

$$\sum_{i=4}^{10} i = (4+5+6+7+8+9+10) = 49$$

$$\sum_{i=1}^{7}(i+3) = ((1+3)+(2+3)+(3+3)+(4+3)+(5+3)+(6+3)+(7+3)) =$$

$$= (4+5+6+7+8+9+10) = 49$$

Subtracting Partial Summation from Total Summation

- The following example illustrates the subtraction technique for dealing with summations where the initial value of the summation range is greater than 1.
- *Example:*
- Notice that: $\sum_{i=4}^{10} i = (4+5+6+7+8+9+10) = 49$
- is equal to: (1+2+3+4+5+6+7+8+9+10) – (1+2+3) = 55 – 6 = 49.
- Another way of stating this is: $\sum_{i=4}^{10} i = \sum_{1}^{10} i - \sum_{1}^{3} i$
- Let’s check to see if this is really correct.

$$\sum_{i=4}^{10} i = \sum_{1}^{10} i - \sum_{1}^{3} i = \frac{10^2+10}{2} - \frac{3^2+3}{2} = \frac{110}{2} - \frac{12}{2} = 55 - 6 = 49$$

Another Example – Range Shifting Technique

- The following example illustrates range shifting technique for dealing with summations where the initial value of the summation range is greater than 1.
- *Example:* Find the value of the summation $\sum_{i=23}^{67}(3i+5)$

To convert $\sum_{i=23}^{67}(3i+5)$ into a summation which starts at 1, shift the index range down by 22.

This means that we have to add 22 to each ***i*** term in the summation. But, we don't add 22 to the 5's because the shifted summation will still have the same number of 5's (45 of them). Shifting the range we have:

$$\sum_{i=23}^{67}(3i+5)=\sum_{i=1}^{45}(3(i+22)+5)=\sum_{i=1}^{45}(3i+66+5)=\sum_{i=1}^{45}(3i+71)$$

Solving this summation we have:

$$\sum_{i=1}^{45}(3i+71)=3\sum_{i=1}^{45}i+71\sum_{i=1}^{45}1=\frac{3(45^2+45)}{2}+71\times 45=\frac{6210}{2}+3195=6300$$

Another Example – Summation Subtraction Technique

- The following example illustrates the subtraction technique for dealing with summations where the initial value of the summation range is greater than 1.
- *Example:* Find the value of the summation $\sum_{i=23}^{67}(3i+5)$

To convert $\sum_{i=23}^{67}(3i+5)$ into a summation which starts at 1, and subtract $\sum_{i=1}^{22}(3i+5)$

$$\sum_{i=23}^{67}(3i+5) = \sum_{i=1}^{67}(3i+5) - \sum_{i=1}^{22}(3i+5) = \left(3\sum_{i=1}^{67} i + 5\sum_{i=1}^{67} 1\right) - \left(\sum_{i=1}^{22} 3i + 5\sum_{i=1}^{22} 1\right)$$

Solving this expression we have:

$$\left(3\sum_{i=1}^{67} i + 5\sum_{i=1}^{67} 1\right) - \left(\sum_{i=1}^{22} 3i + 5\sum_{i=1}^{22} 1\right) = \frac{3(67(68))}{2} + 5(67) - \frac{3(22(23))}{2} - 5(22)$$

$$= 3(67)34 + 5(67) - 3(11)23 - 5(22) = 6834 + 335 - 759 - 110 = 6300$$

Permutations

- A permutation of a set of n objects is simply an ordered arrangement of those objects.
- A common example of permutations deals with words. How many "words" can you form using the letters A, B, C, and D, without repeating any letters.
- Essentially, the question boils down to how many ways you can order a set of objects.
- In this case we know that we have four choices for the first letter. Once that choice is made we have three choices for the second letter, and so on.
  - In particular we have: $4 \times 3 \times 2 \times 1 = 24$ possible permutations
  - ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
  - BACD, BADC, BCDA, BCAD, BDAC, BDCA
  - CABD, CADB, CBAD, CBDA, CDAB, CDBA
  - DABC, DACB, DBAC, DBCA, DCAB, DCBA

Permutations (cont.)

- In general, the number of permutations of *n* objects is:
  - $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!$ permutations
- Now consider the following problem: Given *n* distinct objects, how many permutation are there of *r* of those objects, where $1 \leq r \leq n$?
  - $n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = n!/(n-r)!$ permutations

Combinations

- Notice that when dealing with permutations, the order of the chosen objects matters.
- If the order of the objects does matter, then we are dealing not with a permutation but rather a combination of the objects.
- A combination of *r* objects from a set of *n* objects is a selection of *r* objects without regard to order.
  - For example, if we have five objects: {a, b, c, d, e}, we can choose three of these objects in 10 different ways if order is not taken into account:

    abc abd abe acd ace ade bcd bce bde cde

- In general: $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$

Combinations (cont.)

- The values $\binom{n}{r}$ are called the binomial coefficients because of the role they play in what is commonly referred to as Newton's Theorem.
- Newton's Theorem

$$\boxed{(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n}$$

- Note: if x = 1 we have: $\sum_{r=0}^{n} \binom{n}{r} = 2^n$
- A table displaying the values of $\binom{n}{r}$ is often called Pascal's triangle an is shown on the next page.

Pascal's Triangle

n/r	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

$$\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{5040}{4! \times 3!} = \frac{5040}{144} = 35$$