Notes for Recitation on Sums (Lectures/Practice Problems)

A summation is simply a list of numbers to add. Here is an example of a sum written out:

5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29

While I was able to write out these 13 values individually, if I wanted to add a million values, it would not be feasible to write out each individually by hand. Since adding many numbers is such a powerful and common tool in mathematics, we must create a short-hand notation system, so that we can write less, but still unambiguously confer the exact meaning of the values to be added. We can express what is written above as follows:

$$\sum_{k=2}^{14} (2\mathrm{k} + 1)$$

Each summation uses the Greek capital letter Sigma. Most summations (not all) define a "variable" typically referred to as the summation index in the lower left corner of the sigma sign, and have that set equal to a value of some sort. In this summation that value is 2, but the value can be in terms of a variable as well. On the top of the sigma sign is another value, in this case, 14, but that can also be in terms of some variable instead of being a fixed number. To the right of the sigma sign is the function (in terms of the summation index) that we are going to be adding. To "expand" what this summation means, do the following:

1) Start a running tally to 0 (this is your sum).

2) Plug in k = 2 (the first shown value below the sigma sign) into the function to the right of the sigma sign. Add this to the running tally.

3) Set k (the summation index) to the next integer value greater than the previous value of k. In short, add 1 to the old value of k.

4) Plug in this new value for k into the function to the right of the sigma sign and add this to the running tally.

Future Steps: Continue in this fashion until you plug in k = 14 (the top value) into the function to the right of the sigma sign and add that into the running tally. At this point stop and the summation is whatever the running tally equals.

So, in this case, the summation above, when expanded is:

[2(2) + 1] + [2(3) + 1] + [2(4) + 1] + … + [2(14) + 1]

As an exercise, expand out these three summations (note the use of j in the last sum is NOT a typo!)

$$\sum_{a=10}^{15} (3a - 2)$$

$$\sum_{k=4}^{7} k2^k$$

$$\sum_{j=100}^{105} 3k$$

While the last one seems tricky, what we see is that as j changes the function f(j) = 3k, does NOT change. Thus, even if something has a letter in it, it may be a constant, with respect to a particular summation index. For any constant c, which doesn't change as a summation index k changes, we get the following formula:

$$\sum_{k=a}^{b} c = c(b - a + 1)$$

In English, what this says is that if we add the number c once for each integer in between a and b, we'll get c times (b - a + 1). The +1 is because from integer a to integer b inclusive, as long as $b \geq a$, we have b - a + 1 integers. Just try it out for yourself. Count from 5 to 8: 5, 6, 7, 8. There are four integers, not 3.

Let's apply this formula a few times, try for yourself:

$$\sum_{k=101}^{200} 47$$

$$\sum_{k=n+1}^{2n} n^2$$

$$\sum_{k=n}^{n^2} 7n$$

Now, let's look at a summation where the function inside of the summation changes value as the summation index changes value. One example is the following formula, which will be proved later:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

This just says that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. Let me apply this summation a couple times as an example:

$$\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050$$

$$\sum_{k=1}^{4n+7} k = \frac{(4n+7)((4n+7)+1)}{2} = \frac{(4n+7)(4n+8)}{2} = (2n+4)(4n+7)$$

In the first example, 100 is in the place of n, so everywhere we see n in the right hand side, we replace it with 100. In the second example, everywhere we see n in the RHS, we replace it with 4n+7. This may sound confusing for sum, but keep in mind that for the original formula, we could have used a different variable than n, say z or something and what we are really doing then, is plugging in z = 4n + 7 in each slot there is a z. So, one trick that may help you is to always write the generic formula with a letter that is different than all the letters in the problem you are doing. Finally, you must do the substitution in parentheses, since the n in the original formula is a single thing, so to speak.

Here are some examples for you to try:

$$\sum_{k=1}^{2n} k$$

$$\sum_{i=1}^{2^n} i$$

$$\sum_{k=1}^{a+b} k$$