

CIS 3362 - 10/27/23

1972 Clifford Cocks - mathematician (IMO)

In 3 weeks he came up with RSA, followed by  
Diffie-Hellman. Declassified 1997

Alternatives to RSA...

potential improvements

- 1) fast mod expo slow compared very often with fact.
- 2) If factoring ever becomes "easier" / "faster" we need a new alternative that doesn't get compromised in that situation.

① El Gamal

② Elliptic Curve Cryptography \* (where we'll move)  
(to probably)

Public Elements

large prime =  $q$   
primitive root =  $\alpha$

Alice's Public Keys

Alice does this:  
~~We want to send a message to Alice~~

- 1) Pick random int  $X_A$   $1 \leq X_A \leq q-1$ .
- 2)  $Y_A = \alpha^{X_A} \pmod{q}$  (Alice's Public Key)

→ Alice's Secret Key

We want to send a msg to Alice:

1.  $M$  - plaintext  $0 \leq M \leq q-1$

2. Bob/Whoever choose and int  $k$ ,  $1 \leq k \leq q-1$

3. Bob computes  $K = Y_A^k \pmod{q}$ . (Just like Diffie-Hellman)

4. Ciphertext Bob sends  $(C_1, C_2)$

$$\begin{cases} C_1 = 2^k \pmod{q} & \text{(Just like Diffie-Hellman)} \\ C_2 = KM \pmod{q} \end{cases}$$

How does Alice Decrypt?

1.  $C_1^{x_A} = (2^k)^{x_A} = 2^{kx_A} = Y_A^k = K$   
(Takes  $C_1$  raises power  $x_A$ .)  
This yields  $k$ .

2. Compute  $k^{-1} \pmod{q}$  via E&F.

3.  $M = (k^{-1}C_2) = \underline{(k^{-1}KM)} = M \pmod{q}$

for each choice of  $k$ , we produce a diff  $C_1, C_2$

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graph LR; M --> C11["C11, C21"]; M --> C12["C12, C22"]; M --> C13["C13, C23"]
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Slower than RSA

## El Gamal Cryptosystem

The El Gamal Cryptosystem uses the same two global elements that the Diffie-Hellman Key Exchange uses, a large prime number which we'll call  $q$ , and a primitive root, which we'll call  $\alpha$ .

The user Alice (A), to whom messages can be sent does the following:

1. Generates a random integer  $X_A$  such that  $1 < X_A < q - 1$ . (This is her private key.)
2. Computes  $Y_A = \alpha^{X_A} \pmod{q}$ . (This is her posted public key others use to send messages to her.)

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Here is how Bob (or anyone for that matter), can send an encrypted message to Alice:

1. M, the plaintext message must be an integer in between 0 and  $q-1$ , inclusive.
2. Bob chooses a random integer,  $k$ ,  $1 \leq k \leq q - 1$ . (Note: Each choice of  $k$  he selects results in a different ciphertext. Thus, for this cipher, a single plaintext maps to many possible different ciphertexts, and each of those ciphertexts maps back to the same plaintext. The value of  $k$  that Bob selects determines WHICH of the possible ciphertexts for  $M$  will get created.)
3. Compute  $K = Y_A^k \pmod{q}$ . This is a piece used to create a portion of the ciphertext.
4. The ciphertext is the ordered pair  $(C_1, C_2)$ , computed as follows:

$$C_1 = \alpha^k \pmod{q} \quad C_2 = KM \pmod{q}$$

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Here is how Alice recovers the plaintext:

1. She needs to figure out Bob's value of  $K$  and can do so as follows:

$$K = C_1^{X_A} \pmod{q} \quad (\text{If you look carefully, this is precisely Diffie-Hellman!})$$

2. She computes  $K^{-1} \pmod{q}$  via the Extended Euclidean Algorithm.
3. She recovers the message by calculating  $(C_2 K^{-1}) \pmod{q}$ .

First, let's prove she recovers the message from Bob:

Recall that  $C_1 = \alpha^k \pmod{q}$  and  $K = Y_A^k \equiv (\alpha^{X_A})^k \equiv \alpha^{kX_A} \pmod{q}$ .

Now, she calculates  $C_1^{X_A} \equiv (\alpha^k)^{X_A} \equiv \alpha^{kX_A} \pmod{q} = K$ , as shown above.

Once she knows  $K$ , as previously discussed, she can obtain  $K^{-1}$ , and it should be fairly clear that

$C_2 K^{-1} \equiv (KM)K^{-1} \equiv (KK^{-1})M \equiv M \pmod{q}$ , thus, Alice properly recovers the plaintext.

The key drawback to this method is that the size of the ciphertext is twice the size of the plaintext, and the time used in the process is slightly more (a couple steps added to a modular exponentiation for both encryption and decryption compared to RSA).

Let's see an example with small values, so we can see the effect of picking different values of  $k$  for encryption.

Let's choose  $q = 11$  and  $\alpha = 2$ . Let Alice choose  $X_A = 4$ , so  $Y_A = 2^4 \bmod 11 = 5$ .

Here is a chart of 2 raised to each power mod 11:

Exp	0	1	2	3	4	5	6	7	8	9	10
Value	1	2	4	8	5	10	9	7	3	6	1

Here is a chart of all possible ways to encrypt  $M = 7$ , selecting values of  $k$  from 1 to 10:

M	k	K	$C_1$	$C_2$	$C_1^{X_A}$	$K^{-1}$	$K^{-1}C_2$
7	1	5	2	2	5	9	7
7	2	3	4	10	3	4	7
7	3	4	8	6	4	3	7
7	4	9	5	8	9	5	7
7	5	1	10	7	1	1	7
7	6	5	9	2	5	9	7
7	7	3	7	10	3	4	7
7	8	4	3	6	4	3	7
7	9	9	6	8	9	5	7
7	10	1	1	7	1	1	7

Thus, each of the ten ordered pairs  $(2, 2), (4, 10), (8, 6), (5, 8), (10, 7), (9, 2), (7, 10), (3, 6), (6, 8)$  and  $(1, 7)$  are all possible encryptions of the plaintext  $M = 7$ .

Bob's  
Random  
choice

Result of Diffie-Hellman

$$K = (2^{X_A})^k$$