1) (6 pts) Determine the Prime Factorization of 121,564,800.

\[ 121564800 = 1215648 \times 10^2, \text{ use calculator to continue dividing out copies of 2} \]
\[ = 2^5 \times 37989 \times 2^2 \times 5^2, \text{ use calculator to divide out copies of 3} \]
\[ = 2^7 \times 3^4 \times 5^2 \times 469, \text{ now divide out 7 to get the final answer} \]
\[ = 2^7 \times 3^4 \times 5^2 \times 7^1 \times 67^1 \]

\[ 2^7 \times 3^4 \times 5^2 \times 7^1 \times 67^1 \]

Grading: 1 pt per term (term has to be completely correct to get the point), 1 pt bonus for getting all 5 terms.

2) (6 pts) Determine \( \varphi(121564800) \) and give your answer in prime factorized form.

\[ \varphi(121564800) = \varphi(2^7 \times 3^4 \times 5^2 \times 7^1 \times 67^1) = \]
\[ = \varphi(2^7) \times \varphi(3^4) \times \varphi(5^2) \times \varphi(7^1) \times \varphi(67^1) \]
\[ = (2^7 - 2^6)(3^4 - 3^3)(5^2 - 5^1)(7 - 1)(67 - 1) \]
\[ = 2^6 \times 2 \times 3^3 \times 2^2 \times 5 \times 2 \times 3 \times 2 \times 3 \times 11 \]
\[ = 2^{11} \times 3^5 \times 5^1 \times 11^1 \]

\[ 2^{11} \times 3^5 \times 5^1 \times 11^1 \]

Grading: 2 pts to list each term in the product via definition of phi.
2 pts to simplify to an answer that isn’t prime factorized (27371520)
2 pts to prime factorize. (Note, no need to get to intermediate answer above to get full credit.)
3) (6 pts) Determine the remainder when $11^{2187}$ is divided by 313. Note that 313 is prime. **For full credit use Fermat’s Theorem.**

By Fermat’s Theorem, $11^{312} \equiv 1 \pmod{313}$. Let’s look at the given expression:

$$11^{2187} = 11^{2184} \times 11^3 = (11^{312})^7 \times 11^3 \equiv 1^7 \times 1331 \equiv 79 \pmod{313}.$$ 

It follows that the desired remainder is **79**.

**Grading:** 2 pts for stating Fermat’s theorem or properly applying it.  
2 pts for proper exponential breakdown to get ready to plug in Fermat’s.  
2 pts to reduce to final answer.

4) (6 pts) Determine the remainder when $2681^{96002}$ is divided by 20200. **For full credit use Euler’s Theorem.**

$$20200 = 100 \times 202 = 2^2 \times 5^2 \times 2 \times 101 = 2^3 \times 5^2 \times 101$$

$$\varphi(20200) = \varphi(2^3) \times \varphi(5^2) \times \varphi(101^1) = (2^3 - 2^2)(5^2 - 5^1)(101 - 1) = 4 \times 20 \times 100 = 8000.$$ 

It follows by Euler’s Theorem, since $\gcd(2681, 20200) = 1$, $2681^{8000} \equiv 1 \pmod{20200}$

$$2681^{96002} = 2681^{96000} \times 2681^2 = (2681^{8000})^{12} \times 2681^2 \equiv 1^{12} \times 7187761 \equiv 16761 \pmod{20200},$$ 

use the calculator to reduce the mod.

It follows that the desired remainder is **16761**.

**Grading:** 1 pt prime fact, 1 pt phi, 1 pt stating Euler’s, 1 pt expo breakdown, 2 pts reduce to final answer.
5) (8 pts) Use the Fermat Factoring Method to factor 91787. Please fill out the table below. Note: More rows than necessary are provided.

<table>
<thead>
<tr>
<th>x</th>
<th>x^2 - 91787</th>
<th>Perfect Square?</th>
</tr>
</thead>
<tbody>
<tr>
<td>303</td>
<td>22</td>
<td>No</td>
</tr>
<tr>
<td>304</td>
<td>629</td>
<td>No</td>
</tr>
<tr>
<td>305</td>
<td>1238</td>
<td>No</td>
</tr>
<tr>
<td>306</td>
<td>1849 = 43 x 43</td>
<td>Yes</td>
</tr>
</tbody>
</table>

91787 = (306+43) x (306-43) = 349 x 263

**Grading: 1 pt first row, 2 pts second row, 2 pts third row, 2 pts third row, 1 pt final answer**

6) (10 pts) The multiplicative order of an integer $a$ modulo $n$, is the smallest positive integer $k$, such that $a^k \equiv 1 \pmod{n}$. (Note: the term is only defined for values of $a$ such that gcd$(a, n) = 1$.) Let $f_p(k)$ equal the number of integers in the range $[1,p-1]$ which have multiplicative order $k$. Determine the following values: $f_{101}(10)$, $f_{101}(20)$, $f_{101}(25)$, $f_{101}(50)$, and $f_{101}(100)$. (Note: full credit will only be given if students show proper justification of their answers AND use an efficient method of solution.)

What we proved in homework 5 is that there are exactly $\varphi(k)$ bases which have order $k$ modulo a prime, $p$. It follows that the desired answers are:

- $\varphi(10) = (2-1)(5-1) = 4$
- $\varphi(20) = (2^2-2^1)(5-1) = 8$
- $\varphi(25) = (5^2-5^1) = 20$
- $\varphi(50) = (2-1)(5^2-5^1) = 20$
- $\varphi(100) = (2^2-2^1)(5^2-5^1) = 40$

$f_{101}(10) = 4$, $f_{101}(20) = 8$, $f_{101}(25) = 20$, $f_{101}(50) = 20$, $f_{101}(100) = 40$

**Grading: 5 pts for stating the result from the homework,**
**1 pt each for properly applying the result.**
**If there’s no work, or no proper justification 0/10**
7) (7 pts) Write a function that takes in a positive integer, \( n \), and returns the sum of divisors of \( n \). If the run time of your function is \( O(n) \) you will get 3 points out of 7. For full credit, your run time must be \( O(\sqrt{n}) \). It is guaranteed that the input value \( n \) does not exceed \( 10^{12} \). (This restriction ensures that the result fits in a long long.)

```cpp
long long sumDivisors(long long n) {
    long long res = 0; // 1 pt
    for (long long i=1; i*i<=n; i++) { // 1 pt loop
        if (n%i == 0) { // 1 pts
            res += i; // 1 pt
            if (n/i > i) // 1 pt
                res += n/i; // 1 pt
        }
    }
    return res; // 1 pt
}
```

**Grading note:** 3 pts out of 7 for \( O(n) \) code

- Only use criteria above if code is fast enough
- Give 5 out of 7 if \( n/i \) is added in all cases

8) (1 pts) What animal is used for the logo of fast food chain Panda Express?

**Panda, Give to all**