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On Uniqueness of Solutions of the Three-Light-Source Photometric Stereo: Conditions on Illumination Configuration and Surface Reflectance

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This paper is concerned with photometric methods using three images with different lighting direction to obtain shape information of an object. Such methods are based on the photometric equation that relates the normal of the object surface to the triplet of the image brightness. This paper discusses the issue of whether the surface normal and the orientation of the 3-vector formed by the image brightness triplet is one-to-one in the equation. Several types of photometric methods require this relation to be one-to-one. We mainly consider the case where the reflectance map is an increasing function of the angle between the surface normal and the illuminant direction. We first point out that even in this simple case, it is possible that the relation is not one-to-one. Then we derive several sufficient conditions on the reflectance as well as the illumination configuration for the one-to-one relation.

1. INTRODUCTION

Photometric stereo is a method of recovering the surface of an object from its multiple images taken from a fixed viewpoint under different illuminant directions. Since it was developed by Woodham [1], a large number of studies have been conducted on the method and various kinds of extension have been made. Among them, remarkable is the development of the methods that are able to derive some valuable information about the object shape with only partial knowledge of illumination conditions and the surface reflectance [2–5].

One such development is the methods of computing the sign of the Gaussian curvature (we will refer to it as SGC) on the object surface [6, 7]. These methods do not require knowledge of illumination conditions or knowledge of the surface reflectance. In the literature the authors claimed that the methods were applicable to object surfaces of non-Lambertian

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diffuse reflectance. However, there is no guarantee that these methods work well for all of diffuse reflectance, and this needs to be explored furthermore.

All photometric methods that use three images taken under different illuminant directions are based on the three-light-source photometric equation. The equation relates the normal of the surface to the triplet of the image brightness. This relation can be expressed in several ways. Among them, this paper considers the relation between the surface normal and the *orientation* of the 3-vector formed by the triplet. As described in [6, 7], in order to compute SGC, this relation must be one-to-one. This is also a condition for photometric stereo without a priori knowledge of light source strength to have a unique solution.

The aim of this paper is to derive conditions on the surface reflectance as well as the illumination for the above relation to be one-to-one. In the Lambertian case, it is simply expressed: the illuminant directions should be linearly independent. In the non-Lambertian case, however, we encounter some difficulties, since we must consider a wide variety of surface reflectance. We need a versatile model that represents most of them well, but such a model would be too complicated. As a result, it would not provide useful information.

Tagare and deFigueiredo tackled this problem and discussed uniqueness of the solution of photometric stereo [8]. On uniqueness of the photometric stereo without knowledge of light source strength, they derived a simple condition on the reflectance map. They claimed that the derived condition could be used for checking the surface reflectance of the target object and the illumination conditions used for the image acquisition. However, their result is not sufficient when one wants to know how to set the illuminant directions so that the solution becomes unique. Also, it is not sufficient when one wants to study the nature of the surface reflectance such that the solution becomes unique. This paper aims to obtain useful results such that these requirements are fulfilled.

2. THE PROBLEM

This section defines the problem considered in this paper.

2.1. Relation between Surface Normal and Triplet of the Image Brightness

We take three images $I_1(x, y)$, $I_2(x, y)$, and $I_3(x, y)$ from a fixed viewpoint by changing illuminant direction. Let $\mathbf{I} \equiv [I_1, I_2, I_3]^\top$ denote a triplet of the image brightness at an image point (x, y). We assume orthographic projection and let $z(x, y)$ denote the object surface. We denote the gradient of the surface by $(p, q) = (\partial z/\partial x, \partial z/\partial y)$. Then the normal of the surface is written by $\hat{\mathbf{n}} = [p, q, 1]^\top/\sqrt{1 + p^2 + q^2}$.

By neglecting interreflection and shadow on the surface, we may represent the image brightness using the reflectance map [9]. Then a change in the illumination is fully expressed by a change of the reflectance map. We let $R_k(p, q)$ or $R_k(\hat{\mathbf{n}})$ denote the kth reflectance map for the kth image ($k = 1, 2, 3$). Then the kth image is written as

$$I_k(x, y) = \rho(x, y) R_k(\hat{\mathbf{n}}) \tag{1a}$$

$$= \rho(x, y) R_k(p, q). \tag{1b}$$

Here, $\rho(x, y)$ is a component of the reflectance inherent to the object surface, called an albedo in the Lambertian reflectance. We allow $\rho(x, y)$ to vary across the object surface.

Using the reflectance map, it can be seen that the triplet of the image brightness **I** has the following relation to the surface normal $\hat{\mathbf{n}}$:

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \rho \begin{bmatrix} R_1(\hat{\mathbf{n}}) \\ R_2(\hat{\mathbf{n}}) \\ R_3(\hat{\mathbf{n}}) \end{bmatrix}. \tag{2}$$

This relation is viewed as a mapping from $\hat{\mathbf{n}}$ to **I**. We denote this mapping by Ψ. We derive another representation of mapping from this relation, which is a mapping from $\hat{\mathbf{n}}$ to the *orientation* of **I**, a mapping from a 2D space onto a 2D space. For example, it can be expressed as

$$(\theta_n, \phi_n) \mapsto (\theta_I, \phi_I),$$

where θ_n and ϕ_n are the zenith and azimuth angles of $\hat{\mathbf{n}}$, and θ_I and ϕ_I are those of **I**. We denote this mapping by Φ.

2.2. One-to-One Relation of the Mapping

The aim of this paper is to derive the conditions for Φ to be one-to-one. More specifically, we consider the surface reflectance and the illumination condition that make Φ become one-to-one.

In [8], Tagare and deFigueiredo discussed the invertibility of Ψ as well as Φ. (The invertibility and the one-to-one relation are equivalent in this case.) They showed that Φ is invertible if and only if

$$\det \begin{bmatrix} R_1 & R_{1p} & R_{1q} \\ R_2 & R_{2p} & R_{2q} \\ R_3 & R_{3p} & R_{3q} \end{bmatrix} \neq 0. \tag{3}$$

Here, R_{1p} represents $\partial R_1/\partial p$. (In [8], they used the zenith and azimuth angles θ and ϕ to represent the surface normal, and used $R(\theta, \phi)$ to denote the reflectance map. Hence, in their original representation of (3), $R_{1\theta}$ and $R_{1\phi}$, etc. are used instead of R_{1p} and R_{1q}, etc., but their results are basically equivalent to (3).) Then they claimed that the invertibility of Φ can be checked by (3).

The condition (3) is, however, not convenient when one wants to know how to arrange the illuminant directions so that Φ becomes one-to-one, or wants to study the nature of the surface reflectance such that Φ is one-to-one. This is because (3) is in a general form, and is not expressed in terms of the illuminant directions or some practical parameters of the reflectance. In this point, we explore the problem in more detail.

First, we exclude the specular reflection and consider only the diffuse reflection. If the surface reflectance has a specular component, then Φ is usually not one-to-one except in some trivial cases. Hence, we restrict our attention to the diffuse reflectance. There are still all sorts of diffuse reflectance, however. It is not easy to represent all of them by a specific model, and if possible, the model would be too complicated. As a result, it will not provide any practical information.

For these reasons, we mainly study the case where the reflectance map is written by $R(\hat{\mathbf{n}}) = \rho f(\hat{\mathbf{l}}^\top \hat{\mathbf{n}})$. (Here, $\hat{\mathbf{l}}$ is the illuminant direction.) This is sometimes called the generalized Lambertian model when f is increasing. Probably, this model is too simple and not

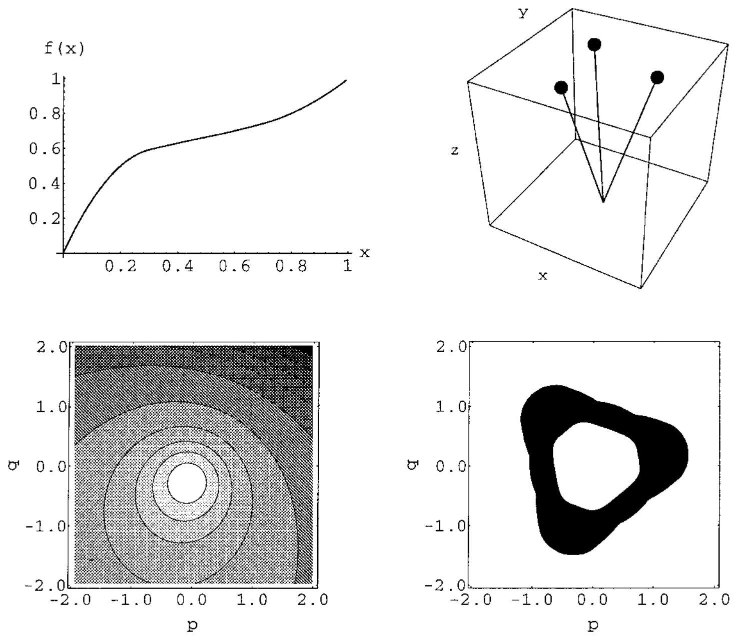

FIG. 1. An example of the mapping Φ that is not one-to-one when f is increasing. (upper left) f. (upper right) Three illuminant directions. (lower left) One of the three reflectance maps. (lower right) Distribution of the sign of the determinant of (3) over the gradient space; a point of positive sign is in white and a point of negative sign is in black. If Φ is one-to-one, it should be either positive everywhere or negative everywhere.

many real reflectances can be well represented by this model. It should be noted, however, that even when f is increasing, it is possible that Φ is not one-to-one. An example is shown in Fig. 1. In this example, f is assumed to be an increasing function shown in the upper left of Fig. 1. An instance of its associated reflectance map for some illuminant direction is shown in the lower left of Fig. 1. The lower right of Fig. 1 shows sign distribution of the determinant on the left-hand side of Eq. (3), when the three illuminant directions are arranged as shown in the upper right of Fig. 1. The region of negative sign is in black and that of positive sign is in white. It can be seen that the determinant of Eq. (3) takes both a positive value and a negative value. On the boundaries of the two regions, the determinant becomes zero and the condition (3) breaks down. This means that several photometric methods, for example, a method of photometric stereo and the method of computing SGC, would yield erroneous results in this case. Hence, even this simple case of generalized Lambertian reflectance needs to be examined.

3. APPLICATIONS

Before discussing the stated problem, we briefly summarize several applications of the results that we will obtain in the next section.

3.1. Unnormalized Phhotometric Stereo

There are two classes of photometric stereo methods whose basic principle is derived from Eq. (2). One is called the *normalized* photometric stereo, which determines $\hat{\mathbf{n}}$ from $\mathbf{I}$

when $R_k(\cdot)$ and also ρ are given. This is an inverse problem about the mapping we call Ψ. The other is called the *unnormalized* photometric stereo, which determines $\hat{\mathbf{n}}$ from $\mathbf{I}$ when only $R_k(\cdot)$ is given. In this method, ρ is not given, and thus one must determine $\hat{\mathbf{n}}$ and ρ simultaneously, or determine $\hat{\mathbf{n}}$ from the orientation of $\mathbf{I}$. This is an inverse problem about the mapping Φ.

In each problem, the global uniqueness of the solution is dependent on the invertibility of Ψ and Φ. In this paper we discuss the condition for Φ to be one-to-one, and that is equivalent to the condition for Φ to be invertible. Hence, the result that will be obtained is applicable to the unnormalized photometric stereo.

3.2. Computation of Curvature Sign without Knowledge of Illumination

It is possible to compute the SGC on the object surface from three images taken under different illuminant directions [6, 7]. These methods require Φ to be one-to-one. In what follows, we explain this by summarizing why the SGC is computed even when the illuminant directions are unknown. (Note that the derivation below is novel and different from those described in [6, 7].)

We first define a 3×3 matrix $\mathbf{D}$ composed of the triplet of the image brightness and its directional derivatives in x and y:

$$\mathbf{D} = \begin{bmatrix} I_1 & I_{1x} & I_{1y} \\ I_2 & I_{2x} & I_{2y} \\ I_3 & I_{3x} & I_{3y} \end{bmatrix}. \tag{4}$$

Taking the determinant of this matrix and substituting Eq. (1b) into it, we have

$$\det \mathbf{D} = \rho^3 \det \begin{bmatrix} R_1 & R_{1p} & R_{1q} \\ R_2 & R_{2p} & R_{2q} \\ R_3 & R_{3p} & R_{3q} \end{bmatrix} \det \begin{bmatrix} p_x & p_y \\ q_x & q_y \end{bmatrix}. \tag{5}$$

The last determinant $p_x q_y - p_y q_x$ on the right-hand side has the same sign as SGC, since the Gaussian curvature of the surface, K, is given by

$$K = \frac{p_x q_y - p_y q_x}{(1 + p^2 + q^2)^2}. \tag{6}$$

Hence, if the condition (3) holds, then the first determinant concerning $R(p, q)$ on the right-hand side of Eq. (5) does not change its sign over the image. Its sign is usually determined by the rotation orientation of the three illuminant directions. If it is known, the SGC is determined by computing the determinant of $\mathbf{D}$.

As described, the condition (3) is equivalent to the condition for Φ to be one-to-one. The issue of whether Φ is one-to-one is essential for the methods of computing SGC.

3.3. Diffuse Non-Lambertian Reflectance

In this paper the generalized Lambertian model is mainly treated. This section discusses how well the generalized Lambertian model approximates the reflectance of real surfaces and shows how the several results that will be obtained in the next section is connected to the reflectance of real surfaces.

The generalization of the Lambertian model, i.e., using $f(\cos\theta)$ instead of $\cos\theta$, is a way of extending the applicability of the model to a more broad range of diffuse reflection without losing its simplicity. By tuning the function f, it approximates various kinds of diffuse reflections with fair accuracy. Of course, it still has the same limitation as the original Lambertian model that the reflected radiance is independent of the viewing direction. Accounting for the fact that the various types of diffuse reflection cannot be covered by any specific model, however, we argue that the approximation by the generalized Lambertian model has a practical meaning.

Although we are concerned with only diffuse reflection here, there are still various types of diffuse reflection. In order to compare the generalized Lambertian model with some real diffuse reflectances, we take two models of diffuse reflection, the Wolff model [10] and the Oren–Nayar model [11]. Each of them successfully describes the reflectance property of some kinds of real surfaces [12].

Figure 2 shows typical reflectance maps of the Wolff model and the Oren–Nayar model. In the figure, the z axis is parallel to the viewing direction, and the illuminant direction is 10° slanted from z axis. There is a parameter called the index of reflaction in the Wolff model (denoted by ρ and n respectively in [10]), and it is set to 1.7 to computing the reflectance map. Also, there is a parameter called the surface roughness (denoted by σ in [11]) in the Oren–Nayar model, and it is set to 40. These two reflectance maps are compared to that of the Lambertian reflectance shown in Fig. 3. It can be seen that for the Wolff model, the peak of the reflectance map is sharper than that of the Lambertian model and that for the

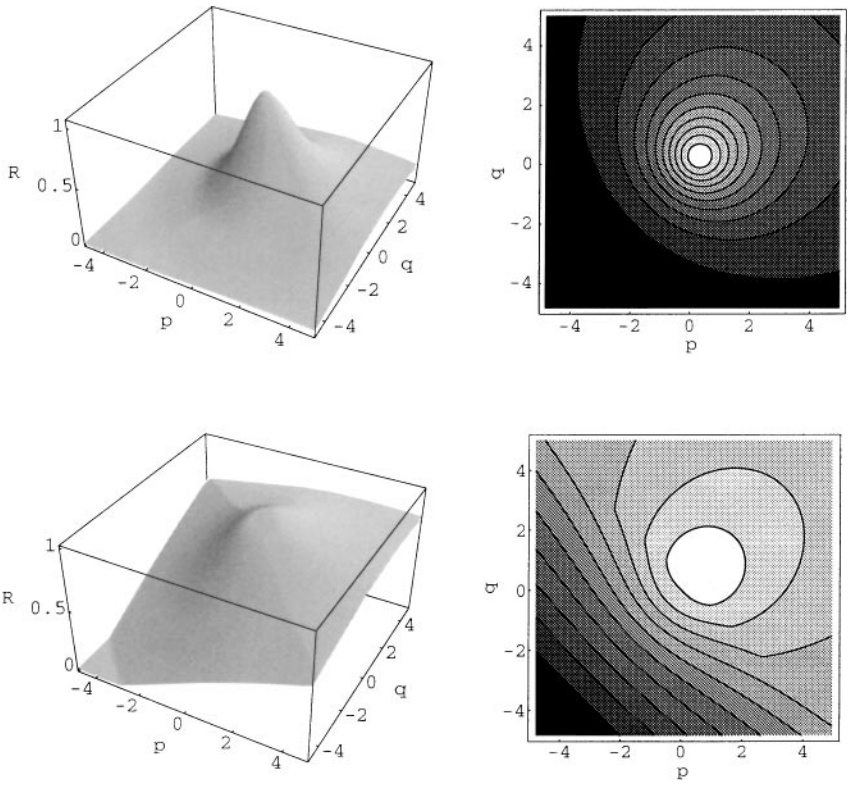

FIG. 2. A reflectance map of the Wolff model (upper row) and a reflectance map of the Oren–Nayar model (lower row).

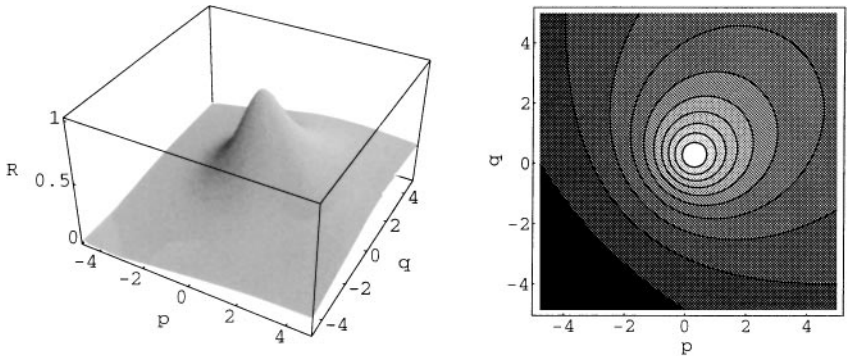

FIG. 3. A reflectance map of the Lambertian model.

Oren–Nayar model, the overall brightness is larger and the decrease rate of the brightness toward the shadow region is smaller.

These two reflectance properties of quite different type can be simulated to a certain extent by the generalized Lambertian reflectance $R(p, q) = f(\hat{\mathbf{l}}^\top \hat{\mathbf{n}})$ by tuning the function *f*. We compute *f* so that the resulting reflectance map is closest to the reflectance maps of Fig. 2 in the area of $-5 < p < 5$ and $-5 < q < 5$. The actual computation is done in the discrete domain of p and q. Figure 4 shows the functions that are computed. Figure 5 shows the resulting reflectance maps. As a matter of course, the contour shapes of the obtained map cannot be different from those of the original Lambertian map. It can be seen, however, that the reflectance map approximating the map of the Wolff model has a sharp peak like the true map and that the map for the Oren–Nayar model has a desired property of the brightness decreasing slowly toward the shadow region. As a result of the approximation, the functions have a complicated shape, as shown in Fig 4. For reflectances other than those described by the Wolff model and the Oren–Nayar model, the function shape may vary depending on the nature of the reflectance. What the shape of f should be so that Φ becomes one-to-one will be the main theme of the next section, along with conditions for the illumination configuration.

4. ONE-TO-ONE RELATION

This section discusses under what conditions the mapping Φ is one-to-one. Mainly the generalized Lambertian reflectance is treated here and for the general reflectance, some results are presented.

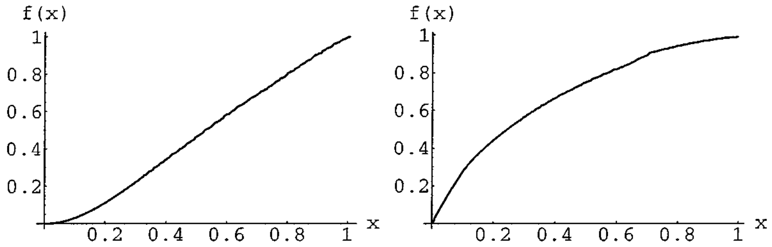

FIG. 4. The functions of the generalized Lambertian reflectance tuned so that the resulting reflectance map approximates the Wolff model (left) and the Oren–Nayar model (right).

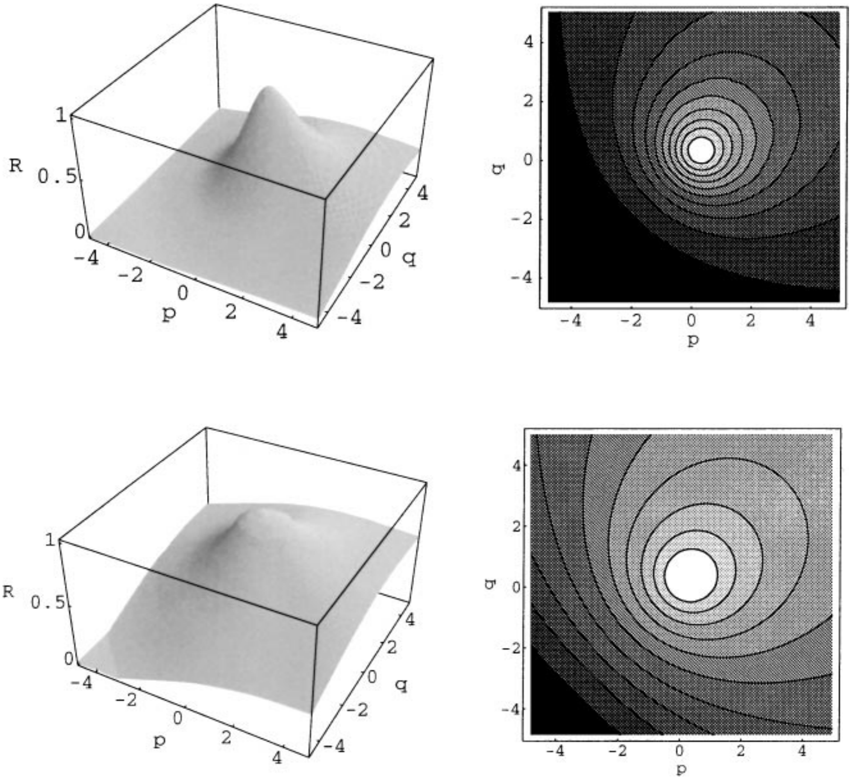

FIG. 5. A reflectance map of the generalized Lambertian model tuned so that it is the closest to the Wolff model (upper row) and the Oren–Nayar model (lower row). Compare with Fig. 2.

4.1. Case of the Generalized Lambertian Reflectance

In the case of generalized Lambertian reflectance, every kth reflectance map is given as $R_k(p, q) = \rho_k f(\hat{\mathbf{l}}_k^\top \hat{\mathbf{n}})$. Here, $\hat{\mathbf{l}}_k$ is the direction of the kth light source, and $f(x)$ is a function defined in [0, 1] such that $f(0) = 0$ and $f(1) = 1$; ρ_k is a product of ρ in Eq. (2) and the illumination strength of the kth light source. We will use the following assumptions:

(H1) f is strictly increasing.
(H2) $\hat{\mathbf{l}}_1$, $\hat{\mathbf{l}}_2$, and $\hat{\mathbf{l}}_3$ are linearly independent.

The triplet of the image brightness is given by

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \rho_1 f(\hat{\mathbf{l}}_1^\top \hat{\mathbf{n}}) \\ \rho_2 f(\hat{\mathbf{l}}_2^\top \hat{\mathbf{n}}) \\ \rho_3 f(\hat{\mathbf{l}}_3^\top \hat{\mathbf{n}}) \end{bmatrix}. \tag{7}$$

Recall that Φ denotes the mapping from $\hat{\mathbf{n}}$ to the *orientation* of $\mathbf{I}$. Recall also that Ψ denotes the mapping from $\hat{\mathbf{n}}$ to $\mathbf{I}$ itself. The domain of both mappings is a set of $\mathbf{n}$ such that $\mathbf{n}^\top \mathbf{n} = 1$.

If (H2) holds, the surface normal $\hat{\mathbf{n}}$ can be uniquely represented in terms of its inner products with $\hat{\mathbf{l}}_1$, $\hat{\mathbf{l}}_2$, and $\hat{\mathbf{l}}_3$. Defining a 3-vector $\mathbf{c} = [c_1, c_2, c_3]^\top = [\hat{\mathbf{l}}_1^T \hat{\mathbf{n}}, \hat{\mathbf{l}}_2^T \hat{\mathbf{n}}, \hat{\mathbf{l}}_3^T \hat{\mathbf{n}}]^\top$ and a 3×3 matrix $\mathbf{L} \equiv [\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2, \hat{\mathbf{l}}_3]^\top$, we have

$$\mathbf{c} = \mathbf{L}\hat{\mathbf{n}}. \tag{8}$$

The assumption (H2) means that $\mathbf{L}$ is nonsingular and $\mathbf{L}^{-1}$ exists. Thus, $\hat{\mathbf{n}} = \mathbf{L}^{-1}\mathbf{c}$. Since

$$|\hat{\mathbf{n}}| = |\mathbf{L}^{-1}\mathbf{c}| = 1, \tag{9}$$

$\mathbf{c}$ is a point on the surface of an ellipsoid in E^3. We assume here that there is no shadow in the image, and restrict $c_k(=\hat{\mathbf{l}}_k^\top \hat{\mathbf{n}})$ to $c_k \geq 0$. Then $\mathbf{c}$ is constrained on a part of the ellipsoid. Let S_c denote this part of the ellipsoid:

$$S_c \equiv \{\mathbf{c} = [c_1, c_2, c_3]^\top \mid |\mathbf{L}^{-1}\mathbf{c}| = 1, c_k \geq 0\}. \tag{10}$$

Using $\mathbf{c} = [c_1, c_2, c_3]^\top$, we may rewrite Eq. (7) as

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} \rho_1 f(c_1) \\ \rho_2 f(c_2) \\ \rho_3 f(c_3) \end{bmatrix}.$$

Since $\hat{\mathbf{n}}$ is uniquely represented by $\mathbf{c}$, we may think of the domain of Ψ as S_c. Letting S_I denote the image of S_c, we may write S_I as

$$S_I \equiv \{\mathbf{I} = [I_1, I_2, I_3]^\top \mid [I_1, I_2, I_3]^\top = [\rho_1 f(c_1), \rho_2 f(c_2), \rho_3 f(c_3)]^\top, \mathbf{c} \in S_c\}. \tag{11}$$

Our objective is to derive a condition for Φ to be one-to-one. This is done by examining the structure of S_I. Based on the photometric equation, one surface normal $\hat{\mathbf{n}}$ yields one triplet $\mathbf{I}$, and therefore one $\hat{\mathbf{n}}$ yields one orientation of $\mathbf{I}$. Thus,Φ is one-to-one if any two distinct normals $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ do not yield $\mathbf{I}_1$ and $\mathbf{I}_2$ such that $\mathbf{I}_1 = \alpha\mathbf{I}_2$. This condition can be stated as follows: $\alpha\mathbf{I} \notin S_I$ for any $\mathbf{I} \in S_I$ and for any $\alpha \neq 1$.

If f is strictly increasing, its inverse f^{-1} exists, which is also strictly increasing, and $f^{-1}(0) = 0$ and $f^{-1}(1) = 1$. Hence, for the mapping Ψ, its inverse Ψ^{-1} exists, which is given by

$$\mathbf{n} = \mathbf{L}^{-1} \begin{bmatrix} f^{-1}(I_1/\rho_1) \\ f^{-1}(I_2/\rho_2) \\ f^{-1}(I_3/\rho_3) \end{bmatrix}.$$

From Eqs. (9), (10), and (11), it can be seen that a triplet $\mathbf{I} = [I_1, I_2, I_3]^\top$ is an element of S_I if and only if

$$\left| \mathbf{L}^{-1} \begin{bmatrix} f^{-1}(I_1/\rho_1) \\ f^{-1}(I_2/\rho_1) \\ f^{-1}(I_3/\rho_3) \end{bmatrix} \right| = 1. \tag{12}$$

We define $T(\alpha)$ as

$$T(\alpha) \equiv \left| \mathbf{L}^{-1} \begin{bmatrix} f^{-1}(\alpha I_1/\rho_1) \\ f^{-1}(\alpha I_2/\rho_2) \\ f^{-1}(\alpha I_3/\rho_3) \end{bmatrix} \right|. \tag{13}$$

(Note that $T(1) = 1$.) Then the condition for Φ to be one-to-one can be expressed as $T(\alpha) \neq 1$ for any $\alpha \neq 1$.

As shown in Fig. 1, the condition that f is increasing does not guarantee Φ to be one-to-one. Intuitively, however, Φ seems to become one-to-one if f has some sort of monotonicity. Such a condition on f actually exists. It is expressed in Theorem 4.1. This requires that the illuminant directions are set symmetrically.

THEOREM 4.1. *Assume (H1) and (H2). Assume also* $\hat{\mathbf{l}}_1$, $\hat{\mathbf{l}}_2$, *and* $\hat{\mathbf{l}}_3$ *are mutually symmetric; that is, the angles between any two of the three vectors are the same. If its first-order derivative is monotonic, that is, its second derivative* $f''(x) \geq 0$ *or* $f''(x) \leq 0$ *for any* $x \in (0, 1)$, *then* Φ *is one-to-one.*

Proof. We first consider the case $f''(x) \leq 0$. Let $g(x) \equiv f^{-1}(x)$. Then $T(\alpha)$ is rewritten as

$$T(\alpha) = |g(\alpha I_1/\rho_1)(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + g(\alpha I_2/\rho_2)(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + g(\alpha I_3/\rho_3)(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|/\det \mathbf{L}. \tag{14}$$

We show that if $\alpha \neq 1$, then $T(\alpha) \neq 1$ by showing $T(\alpha) > 1$ for $\alpha > 1$ and $T(\alpha) < 1$ for $\alpha < 1$. We first show the former. In order to simplify the notations, we define

$$a_i \equiv g(I_i/\rho_i), \quad b_i \equiv g(\alpha I_i/\rho_i), \quad i = 1, 2, 3, \tag{15}$$

and let $T_1(\alpha) \equiv \det \mathbf{L} \cdot T(\alpha)$. Here, we assume $\det \mathbf{L} > 0$. Then we show $T_1(\alpha) > T_1(1)$. Using b_i and a_i, $T_1(\alpha)$ and $T_1(1)$ is written by

$$T_1(\alpha) = |b_1(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + b_2(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + b_3(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|,$$
$$T_1(1) = |a_1(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + a_2(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + a_3(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|.$$

The angles between any two illuminant directions are assumed to be the same. Let c denote their inner products, and

$$c \equiv \hat{\mathbf{l}}_1^\top \hat{\mathbf{l}}_2 = \hat{\mathbf{l}}_2^\top \hat{\mathbf{l}}_3 = \hat{\mathbf{l}}_3^\top \hat{\mathbf{l}}_1. \tag{16}$$

Since $|\hat{\mathbf{l}}_i| = 1$, $-1 < c < 1$. Several vector products can be expressed using c as

$$(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3)^\top (\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) = (\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1)^\top (\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2) = (\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3)^\top (\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2) = c(c-1), \tag{17}$$

and

$$|\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2| = |\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3| = |\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1| = \sqrt{1 - c^2}. \tag{18}$$

Using these, the square of $T_1(\alpha)$ is written as

$$T_1(\alpha)^2 = (1 - c^2)\bigl(b_1^2 + b_2^2 + b_3^2 - 2c\big/(1 + c)(b_1 b_2 + b_2 b_3 + b_3 b_1), \tag{19}$$

and the square of $T_1(1)$ is written as

$$T_1(1)^2 = (1 - c^2)\bigl(a_1^2 + a_2^2 + a_3^2 - 2c\big/(1 + c)(a_1 a_2 + a_2 a_3 + a_3 a_1). \tag{20}$$

We denote $C \equiv -2c/(1+c)$. Since $-1 < c < 1$, $-1 < C$. Taking the difference between $T_1(\alpha)^2$ and $T_1(1)^2$, we have

$$T_1(\alpha)^2 - T_1(1)^2 = (1-c^2)\{b_1^2 + b_2^2 + b_3^2 - a_1^2 - a_2^2 - a_3^2 \\ + C(b_1b_2 + b_2b_3 + b_3b_1 - a_1a_2 - a_2a_3 - a_3a_1)\}. \tag{21}$$

It can be seen from their definition that a_i and b_i satisfy $a_i < b_i$ for $\alpha > 1$, since $g(\cdot)$ is increasing. Thus, it can be shown that $T_1(\alpha) > T_1(1)$ if $C > 0$. Then we consider the case $C \leq 0$. Since $a_i < b_i$, the parenthesized part $b_1b_2 + \cdots - a_3a_1$ in (21) is positive, and thus it is sufficient if it can be shown that $(T_1(\alpha)^2 - T_1(1)^2)/(1-c^2) \geq 0$ for the lower limit $C = -1$. It is reduced as

$$\begin{aligned}&(T_1(\alpha)^2 - T_1(1)^2)/(1-c^2)\\ &\quad = b_1^2 + b_2^2 + b_3^2 - b_1b_2 - b_2b_3 - b_3b_1 - a_1^2 - a_2^2 - a_3^2 + a_1a_2 + a_2a_3 + a_3a_1\\ &\quad = (b_2-b_1)^2 - (a_2-a_1)^2 + (b_3-b_1)(b_3-b_2) - (a_3-a_1)(a_3-a_2).\end{aligned} \tag{22}$$

Here, the monotonicity of $f'(\cdot)$ is used. Since $f''(x) = -g''(f(x))f'(x)/(g'(f(x)))^2$, it holds that $g''(x) \geq 0$ if $f''(x) \leq 0$. Hence, $g'(\alpha x) \geq g'(x)$ for $\alpha > 1$. Thus, we have

$$\alpha g'(\alpha x) \geq g'(x), \tag{23}$$

for $\alpha > 1$. Let x_1 and x_2 be in $[0, 1]$ such that $g(x_1) = a_1$ and $g(x_2) = a_2$. By integrating the above inequality over the interval $[x_1, x_2]$, we have $b_2 - b_1 \geq a_2 - a_1$. Similarly, we have $b_3 - b_1 \geq a_3 - a_1$ and $b_2 - b_1 \geq a_2 - a_1$. It can be seen from these that Eq. (22) must be either zero or positive. Since this is in the case $C = -1$, we have shown $T(\alpha) > T(1) = 1$ for $C > -1$.

It is left to show $T(\alpha) < 1$ for $\alpha < 1$. Considering $g''(x) \geq 0$ again, we have

$$\alpha g'(\alpha x) \leq g'(x). \tag{24}$$

for $\alpha < 1$. By integrating this inequality in the same way as above, we have $b_2 - b_1 \leq a_2 - a_1$ and so on. Thus, we have shown that the form (22) is either zero or negative, and thus $T(\alpha) < 1$ for $\alpha < 1$.

In the case $\det \mathbf{L} < 0$, it must be shown that $T(\alpha) > 1$ for $\alpha < 1$ and $T(\alpha) < 1$ for $\alpha > 1$. This can be done in the similar way.

If $f''(x) \geq 0$, we show that $T(\alpha) > 1$ for $\alpha < 1$ and $T(\alpha) < 1$ for $\alpha > 1$ in the case $\det \mathbf{L} > 0$, and that $T(\alpha) > 1$ for $\alpha > 1$ and $T(\alpha) < 1$ for $\alpha < 1$ in the case $\det \mathbf{L} < 0$. This can be done also in the same way. ■

In the above proof, $\hat{\mathbf{l}}_1$, $\hat{\mathbf{l}}_2$, and $\hat{\mathbf{l}}_3$ are assumed to make the same angle with each other. This is not so critical, namely, even if their angles are slightly different; the above result should hold. In (21), we showed by considering the case $C = -1$. This is the lower limit and is an impossible case (three directions coincide). Hence, it is conjectured that (21) has some margin and this relaxes the assumption.

In the above proof, f' is assumed to be monotonic. The restriction on the arrangement of the illuminant directions assumed in the above can be dissolved by a further assumption on f. When f can be written as a power function, a simple result is derived.

REMARK 4.1. *Assume* (*H2*). *If f is given by* $f(x) = x^r$ *with a positive number r, then* Φ *is one-to-one.*

Proof. The inverse of f is given by $f^{-1}(x) = x^{1/r}$. For the matrix $\mathbf{L} = [\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2, \hat{\mathbf{l}}_3]^\top$, its inverse $\mathbf{L}^{-1}$ is given by

$$\mathbf{L}^{-1} = [\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3, \hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1, \hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2]/\det \mathbf{L}. \tag{25}$$

Thus, $T(\alpha)$ is reduced as

$$\begin{aligned} T(\alpha) &= |f^{-1}(\alpha I_1/\rho_1)(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + f^{-1}(\alpha I_2/\rho_2)(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + f^{-1}(\alpha I_3/\rho_3)(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|/\det \mathbf{L} \\ &= \alpha^{1/r} |f^{-1}(I_1/\rho_1)(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + f^{-1}(I_2/\rho_2)(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + f^{-1}(I_3/\rho_3)(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|/\det \mathbf{L} \\ &= \alpha^{1/r} T(1) = \alpha^{1/r}. \end{aligned} \tag{26}$$

From this, $T(\alpha) \neq 1$ if $\alpha \neq 1$. ∎

Note that this result holds, regardless of the illuminant direction $\mathbf{l}_k$ (as long as they are linearly independent).

When there is no condition other than (H1) and (H2), we can make Φ one-to-one by setting the illuminant directions so that they make a large angle with each other. This is expressed in Theorem 4.2.

THEOREM 4.2. *Assume* (*H1*) *and* (*H2*). *Assume also that any two vectors of* $\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2$, *and* $\hat{\mathbf{l}}_3$ *make angles larger than 90°. Then* Φ *is one-to-one.*

Proof. Since f is increasing, f^{-1} exists and is also increasing. From this we have $f^{-1}(\alpha I_k \rho_k) > f^{-1}(I_k/\rho_k)$ for $\alpha > 1$ and $f^{-1}(\alpha I_k \rho_k) < f^{-1}(I_k/\rho_k)$ for $\alpha < 1$. From the assumption that $\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2$, and $\hat{\mathbf{l}}_3$ make an angle larger than 90°, $\hat{\mathbf{l}}_i^\top \hat{\mathbf{l}}_j < 0$ for $i \neq j$. From this we have

$$(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3)^\top (\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) > 0 \tag{27a}$$

$$(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1)^\top (\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2) > 0 \tag{27b}$$

$$(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)^\top (\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) > 0, \tag{27c}$$

since

$$(\hat{\mathbf{l}}_i \times \hat{\mathbf{l}}_j)^\top (\hat{\mathbf{l}}_j \times \hat{\mathbf{l}}_k) = \left(\hat{\mathbf{l}}_i^\top \hat{\mathbf{l}}_j\right)\left(\hat{\mathbf{l}}_j^\top \hat{\mathbf{l}}_k\right) - \left(\hat{\mathbf{l}}_i^\top \hat{\mathbf{l}}_k\right)\left(\hat{\mathbf{l}}_j^\top \hat{\mathbf{l}}_j\right) > 0, \quad (i \neq j \neq k).$$

(Note that $\hat{\mathbf{l}}_j^\top \hat{\mathbf{l}}_j = 1$.) Thus, if $\alpha > 1$, we reduce $T(\alpha)$ as

$$\begin{aligned} T(\alpha) &= |f^{-1}(\alpha I_1/\rho_1)(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + f^{-1}(\alpha I_2/\rho_2)(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + f^{-1}(\alpha I_3/\rho_3)(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|/\det \mathbf{L} \\ &> |f^{-1}(I_1/\rho_1)(\hat{\mathbf{l}}_2 \times \hat{\mathbf{l}}_3) + f^{-1}(I_2/\rho_2)(\hat{\mathbf{l}}_3 \times \hat{\mathbf{l}}_1) + f^{-1}(I_3/\rho_3)(\hat{\mathbf{l}}_1 \times \hat{\mathbf{l}}_2)|/\det \mathbf{L} \\ &= T(1) = 1. \end{aligned} \tag{28}$$

It can be also shown in the same way that $T(\alpha) < 1$ for $\alpha < 1$. Hence, we have shown that $T(\alpha) \neq 1$ if $\alpha \neq 1$. ∎

4.2. *Case of General Diffuse Reflectance*

We next consider the general case where the reflectance map is not necessarily a function of the angle between the surface normal and the illuminant direction. Although this case is difficult to deal with, a result similar to Theorem 4.2 is derived.

THEOREM 4.3. *Let D be an open region in the gradient space* (pq *space*) *such that for any* $(p, q) \in D$ *the following inequalities hold*:

$$R_{2p}R_{3q} - R_{2q}R_{3p} > 0 \tag{29a}$$

$$R_{3p}R_{1q} - R_{3q}R_{1p} > 0 \tag{29b}$$

$$R_{1p}R_{2q} - R_{1q}R_{2p} > 0. \tag{29c}$$

Φ *is one-to-one for a set of the normals*

$$\{\hat{\mathbf{n}} \mid \hat{\mathbf{n}} = [p, q, 1]^\top / \sqrt{1 + p^2 + q^2}, (p, q) \in D\}.$$

Proof. As described earlier, the condition for Φ to be one-to-one is equivalent to the inequality (3) [8]. We may rewrite the determinant in (3) as the vector triple product

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}^\top \left(\begin{bmatrix} R_{1p} \\ R_{2p} \\ R_{3p} \end{bmatrix} \times \begin{bmatrix} R_{1q} \\ R_{2q} \\ R_{3q} \end{bmatrix} \right). \tag{30}$$

For the vector cross product in this form, it can be seen from inequalities (29) that all the components of the resulting vector are positive in D. Since R_1, R_2, and R_3 are always positive, the above vector triple product is always positive and therefore the inequality (3) holds. Hence, we have shown that Φ is one-to-one in D. ■

Note that the same result holds for the region D' where

$$R_{2p}R_{3q} - R_{2q}R_{3p} < 0 \tag{31a}$$

$$R_{3p}R_{1q} - R_{3q}R_{1p} < 0 \tag{31b}$$

$$R_{1p}R_{2q} - R_{1q}R_{2p} < 0, \tag{31c}$$

since (30) always becomes negative.

The above result has a practical meaning in the case of diffuse reflectance. The following assumptions on $R(p, q)$ usually hold for diffuse reflectance:

(H3) $R(p, q)$ is differentiable everywhere.

(H4) The maximal point of $R(p, q)$ is the only one critical point of $R(p, q)$ (i.e., (p, q) where $R_p = R_q = 0$).

(H5) Any level curve of $R(p, q)$ (i.e., $\{(p, q) \mid R(p, q) = t\}$) in the gradient space is a simple closed curve that is convex.

For different reflectance maps $R_1(p, q)$ and $R_2(p, q)$ satisfying the above assumptions, we consider a set of points (p, q) at which $R_{1p}R_{2q} - R_{1q}R_{2p} = 0$. Geometrically, such points are the points at which a R_1's contour and a R_2's contour come in contact with each

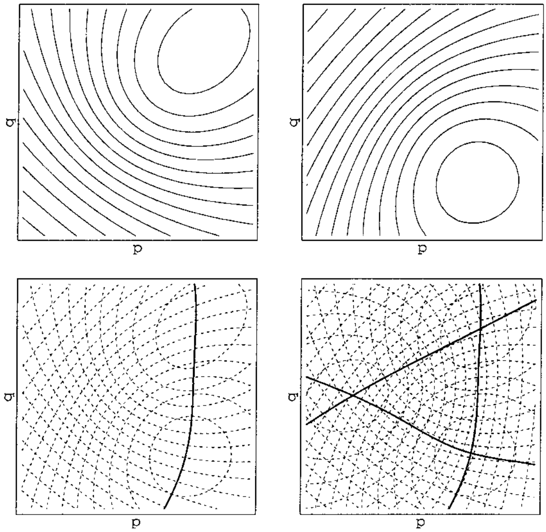

FIG. 6. An example of the region D in Theorem 4.3. (upper left and right) $R_1(p, q)$ and $R_2(p, q)$. (lower left) The line of $R_{1p}R_{2q} - R_{1q}R_{2p} = 0$. (lower right) The region D (the triangular region enclosed by the three curves). The added two lines are drawn by the combination of (R_3, R_1) and that of (R_2, R_3).

other (see Fig. 6). Thus, from the assumption that level curves of $R(p, q)$ are convex, the described set will be a single curve passing through both of the maximum points of R_1 and R_2 as shown in Fig. 6. From the same assumption, the curve will not branch at least within the interval between the two maximal points. This curve divides the gradient space into two regions; on one side of the curve, $R_{1p}R_{2q} - R_{1q}R_{2p}$ is positive, and on the other side, $R_{1p}R_{2q} - R_{1q}R_{2p}$ is negative.

For R_1 and R_3 and also for R_2 and R_3, we have similar boundary curves in the gradient space. Such three boundary curves enclose a region as shown in Fig. 6, which is nothing but D in Theorem 4.3. The above result states that Φ is one-to-one at least in such a region D.

If the above assumptions (H3)–(H5) hold and thus the set of the points of $R_{ip}R_{jq} - R_{iq}R_{jp}\,(i, j = 1, 2, 3)$ forms a single curve, there must be either the region D or D' in the gradient space. Which one appears is dependent on the arrangement of the peaks of $R_i(p, q)\,(i = 1, 2, 3)$ in the gradient space, that is, whether the peaks of R_1, R_2, and R_3 go in a clockwise or counterclockwise sense. Therefore, there always exists a region (D or D') where Φ is one-to-one.

Generally, when we set the illuminant directions so that they make large angles with each other, the maximal points of the reflectance maps move apart from each other, and thus the resulting region D may become large. Although this is desirable, we must set the illuminant directions obliquely with respect to the viewing direction in order to make the angles become large. This usually yields large shadowed regions on the object surface, which may be undesirable in these photometric methods. This is true of Theorem 4.2. Therefore, it is important to balance these two mutually conflicting demands.

5. SUMMARY

We discussed the nature of the three-light-source photometric equation for diffuse non-Lambertian reflectance. The equation relates the orientation of the 3-vector composed of the image brightness to the surface normal. For several photometric methods, whether this relation is one-to-one is an important issue. We derived several sufficient conditions on the surface reflectance and the illuminant directions for that relation to be one-to-one. In the case where the reflectance map is written as $R(\hat{\mathbf{n}}) = \rho f(\hat{\mathbf{n}}^\top \hat{\mathbf{l}})$, we obtained the following results:

- Just because f is strictly increasing and the illuminant directions are linearly independent, it does not follow that the relation is one-to-one.
- If f is a power function with a positive exponent, the above relation is always guaranteed to be one-to-one, so that one may set the illuminant directions arbitrarily as long as they are linearly independent.
- If f's first-order derivative is monotonic, the relation is one-to-one if the illuminant directions are set so that the angles between any two directions are the same.
- If f is increasing, we can make the relation one-to-one by setting the illuminant directions so that they make angles larger than 90°.

Using these results, the illumination should be configured according to the surface reflectance of the target object.

For general diffuse reflectance, we derived the following results:

- There exists a set of the surface normals for which the relation is guaranteed to be one-to-one.
- Generally, we can make the set larger by setting the illuminant directions so that they make larger angles.

The first result just says that at least for the surface normal in a set, the relation is guaranteed to be one-to-one. It does not say anything about the surface normal outside the set.

The above results would give some insight into the problem of planning illumination when using several three-light-source photometric methods such as unnormalized photometric stereo and the methods of computing the curvature sign of the surface. The approximation of real reflectances by the generalized Lambertian model may be sometimes insufficient in terms of the approximation accuracy, and thus further study is required to treat a more wide range of surface reflectances including specular reflection.

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