Vector-Valued Image Regularization with PDE's : A Common Framework for Different Applications

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Abstract

[1] *We address the problem of vector-valued image regularization with variational methods and PDE's. From the study of existing formalisms, we propose a unifying framework based on a very local interpretation of the regularization processes. The resulting equations are then specialized into new regularization PDE's and corresponding numerical schemes that respect the local geometry of vector-valued images. They are finally applied on a wide variety of image processing problems, including color image restoration, inpainting, magnification and flow visualization.*

1. Introduction & Motivation

Anisotropic *regularization PDE's* raise a strong interest in the field of image processing. The ability to smooth data while preserving large global features such as contours and corners (discontinuities), has opened new ways to handle classical image-related issues (restoration, segmentation). Thus, many regularization schemes have been presented so far in the literature, particularly for the case of $2D$ *scalar images* $I : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ ([1, 17, 18, 28] and references therein). Extensions of these algorithms to *vector-valued images* $\mathbf{I} : \Omega \rightarrow \mathbb{R}^n$ have been recently proposed, leading to more elaborated diffusion PDE's : a *coupling between image channels* appears in the equations, through the consideration of a *local vector geometry*, given pointwise by the spectral elements λ_+, λ_- (positive eigenvalues) and θ_+, θ_- (orthogonal eigenvectors) of the 2×2 symmetric and semi positive-definite matrix $\mathbf{G} = \sum_{j=1}^n \nabla I_j \nabla I_j^T$ (also called *structure tensor* [25, 26, 28, 29]). The $\lambda_\pm$ respectively define the local min/max vector-valued variations of $\mathbf{I}$ in corresponding spatial directions $\theta_\pm$, i.e. the local configuration of the image discontinuities. (note that $\lambda_+ = \|\nabla I\|$ and $\theta_+ = \nabla I / \|\nabla I\|$ *for scalar images*, $n = 1$). Regularization schemes generally lie on one of these three following approaches, related to different interpretation levels :

(1) Functional minimization : Regularizing an image $\mathbf{I}$ may be seen as the minimization of a functional $E(\mathbf{I})$ measuring a *global* image variation. The idea is that minimizing this variation will flatten the image, then remove the noise :

$$\min_{\mathbf{I}:\Omega \rightarrow \mathbb{R}^n} E(\mathbf{I}) = \int_\Omega \phi(\mathcal{N}(\mathbf{I}))\, d\Omega \qquad (1)$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function and $\mathcal{N}(\mathbf{I})$ is a norm related to *local image variations*, for instance $\mathcal{N}(\mathbf{I}) = \sqrt{\lambda_+ + \lambda_-} = \mathrm{trace}\,(\mathbf{G})^{\frac{1}{2}}$. The minimization of (1) is performed with a gradient descent (PDE) given by the Euler-Lagrange equations of $E(\mathbf{I})$. Useful references for vector image regularization are [5, 12, 16, 18, 20, 22, 26].

(2) Divergence expressions : A regularization process may be also designed more locally, as the diffusion of pixel values - viewed as chemical concentrations [11, 28] - driven by a 2×2 *diffusion tensor* $\mathbf{D}$ (symmetric and positive matrix) :

$$\frac{\partial I_i}{\partial t} = \mathrm{div}\,(\mathbf{D}\nabla I_i) \qquad (i = 1..n) \qquad (2)$$

It is generally assumed that the spectral elements of $\mathbf{D}$ give the two weights and directions of the local smoothing performed by (2). $\mathbf{D}$ is then usually designed from the spectral elements of $\mathbf{G}$ in order to smooth $\mathbf{I}$ anisotropically, while respecting its intrinsic local geometry by preserving its discontinuities. Anyway, this interpretation of (2) should not be systematic, as pointed out in further paragraphs.

(3) Oriented Laplacians : 2D image regularization may be finally seen as the juxtaposition of two *oriented 1D heat flows*, i.e two monodimensional gaussian smoothing along orthonormal directions $\mathbf{u} \perp \mathbf{v}$, with corresponding weights c_1 and c_2 [14, 19, 25, 26] :

$$\frac{\partial \mathbf{I}}{\partial t} = c_1\, \frac{\partial^2 \mathbf{I}}{\partial \mathbf{u}^2} + c_2\, \frac{\partial^2 \mathbf{I}}{\partial \mathbf{v}^2} = c_1\, I_{\mathbf{uu}} + c_2\, I_{\mathbf{vv}} \qquad (3)$$

Like divergence expressions, c_1, c_2 and $\mathbf{u}, \mathbf{v}$ are usually designed from the spectral elements $\lambda_\pm$ and $\theta_\pm$ of $\mathbf{G}$, in order to perform edge-preserving smoothing, mainly along the direction θ_- orthogonal to the vector image discontinuities.

The link between these three formulations (1),(2),(3) is generally not trivial, especially for vector-valued images. Actually, it is well known for the classical case of ϕ-functional regularization of *scalar* images ($n = 1$). In this case, the three following formulations are equivalent :

$$(1) \quad \min_{I:\Omega \rightarrow \mathbb{R}} \int_\Omega \phi(\|\nabla I\|)\, d\Omega \qquad (4)$$

$$\Rightarrow \quad (2) \quad \frac{\partial I}{\partial t} = \mathrm{div}\left(\frac{\phi'(\|\nabla I\|)}{\|\nabla I\|}\, \nabla I \right)$$

$$\Rightarrow \quad (3) \quad \frac{\partial I}{\partial t} = \frac{\phi'(\|\nabla I\|)}{\|\nabla I\|}\, I_{\xi\xi} + \phi''(\|\nabla I\|)\, I_{\eta\eta}$$

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where $\eta = \nabla I / \|\nabla I\|$ and $\xi \perp \eta$. Note that this regularization leads to *anisotropic smoothing* (in the sense that it is performed in privileged spatial directions ξ and η with different weights), despite the *isotropic shape* of the corresponding divergence-based tensor $\mathbf{D} = \phi'(\|\nabla I\|)/\|\nabla I\| \ \mathbf{Id}$.

In this paper, we propose a way to find such equivalences for the more general case of *vector-valued regularization*. We tackle each of the three interpretation levels (1),(2),(3) in its more general form, and derive the corresponding equations. We particularly show that the oriented-Laplacian formalism has an interesting interpretation in terms of *local filtering*, and represents the right smoothing geometry performed by the PDE's. Then, we design a new vector-valued regularization approach respecting desired local smoothing properties as well as adapted numerical schemes (section 4 and 5). Finally, we apply it for color image restoration, inpainting, magnification, and flow visualization (section 6).

2. From Variational to Divergence Forms

We first consider vector-valued image regularization as a variational problem. We want to find the corresponding *divergence-based expression*, i.e. the link (1)⇒(2).

• **A generic functional** : Instead of considering a functional such as (1) depending on a variation norm $\mathcal{N}(\mathbf{I})$, we rather propose to minimize this more general ψ-functional :

$$\min_{\mathbf{I}:\Omega\to\mathbb{R}^n} E(\mathbf{I}) = \int_\Omega \psi(\lambda_+, \lambda_-)\, d\Omega \qquad (5)$$

where the $\lambda_\pm$ are the eigenvalues of the structure tensor $\mathbf{G} = \sum_{j=1}^n \nabla I_j \nabla I_j^T$, and $\psi : \mathbb{R}^2 \to \mathbb{R}$ is an increasing function. This is a natural and generic extension of the scalar ϕ-function formulation (4) for vector-valued images.

• **Gradient descent** : The Euler-Lagrange equations of (5) can be derived, and reduce to a simple form of divergence-based expression $\frac{\partial I_i}{\partial t} = \mathrm{div}\,(\mathbf{D}\nabla I_i)\,, \quad (i = 1..n)$ (see [27] for a full demonstration), where the 2×2 tensor $\mathbf{D}$ is :

$$\mathbf{D} = \frac{\partial \psi}{\partial \lambda_+}(\lambda_+, \lambda_-)\ \theta_+\theta_+^T + \frac{\partial \psi}{\partial \lambda_-}(\lambda_+, \lambda_-)\ \theta_-\theta_-^T$$

$\mathbf{D}$ is simply defined from the partial derivatives of ψ, and has the same eigenvectors θ_+, θ_- as $\mathbf{G}$.

• **Link with existing approaches** : Particular choices of functions ψ leads to previous vector-valued regularization approaches defined as variational methods, such as the whole range of Vector ϕ-functionals [16, 22] : $\psi(\lambda_+, \lambda_-) = \phi(\sqrt{\lambda_+ + \lambda_-})$, or the Beltrami flow framework [12] : $\psi(\lambda_+, \lambda_-) = \sqrt{(1+\lambda_+)(1+\lambda_-)}$. More generally, our approach shows that eigenvalues of a divergence tensor $\mathbf{D}$ define the *gradient of a potential function* ψ (if such a ψ exists), linked to the functional (5). Anyway, the shape of $\mathbf{D}$ is still giving a wrong estimation of the local smoothing performed by the process : For instance, the ϕ-functional case leads to isotropic tensors $\mathbf{D}$, while the effective local smoothing is anisotropic.

3. From Divergences to Oriented Laplacians

We rather want to develop divergence forms as (2) into their corresponding *oriented Laplacian* formulations, i.e. find the link (2)⇒(3). Indeed, it is particularly understandable in terms of local geometric smoothing :

• **Geometric meaning of oriented Laplacians** : Let us consider the oriented Laplacian-based equation (3). As $\mathbf{u} \perp \mathbf{v}$, this PDE can be equivalently written as :

$$\frac{\partial I_i}{\partial t} = \mathrm{trace}\,(\mathbf{T}\mathbf{H}_i) \qquad (i = 1..n) \qquad (6)$$

where $\mathbf{H}_i$ is the *Hessian matrix of the vector component* I_i and $\mathbf{T}$ is the 2×2 tensor defined as : $\mathbf{T} = c_1\mathbf{u}\mathbf{u}^T + c_2\mathbf{v}\mathbf{v}^T$, characterized by its two eigenvalues c_1, c_2 and its corresponding eigenvectors $\mathbf{u} \perp \mathbf{v}$. Suppose that $\mathbf{T}$ is a constant tensor over the definition domain Ω. Then, it can be shown [24, 27] that the formal solution of the PDE (6) is :

$$I_{i_{(t)}} = I_{i_{(t=0)}} * G^{(\mathbf{T},t)} \qquad (i = 1..n) \qquad (7)$$

where $*$ stands for the convolution operator and $G^{(\mathbf{T},t)}$ is an *oriented gaussian kernel*, defined by :

$$G^{(\mathbf{T},t)}(\mathbf{x}) = \frac{1}{4\pi t} \exp\left(-\frac{\mathbf{x}^T\mathbf{T}^{-1}\mathbf{x}}{4t}\right) \quad \text{with} \quad \mathbf{x} = (x \quad y)^T$$

It is a generalization of the Koenderink's idea [13], who proved this property for the *isotropic diffusion tensor* $\mathbf{T} = \mathbf{Id}$, resulting in the well-known *heat flow* equation : $\frac{\partial I_i}{\partial t} = \Delta I_i$. The top row of Fig.1 illustrates a gaussian kernel $G^{(\mathbf{T},t)}(x, y)$ obtained with an anisotropic tensor $\mathbf{T}$ *(top left)* and the corresponding evolution of the PDE (6) on a color image *(top right)*. It is worth to notice that the representation of $\mathbf{G}^{(\mathbf{T},t)}$ gives *exactly* the classical ellipsoid drawing of $\mathbf{T}$. Conversely, it is clear that $\mathbf{T}$ represents the effective smoothing performed by the PDE (6).

Figure 1: *Behavior of trace-based PDE's (6) (right) with constant or spatially varying tensors* $\mathbf{T}$ *(respectively top and bottom left).*

When $\mathbf{T}$ is not constant (which is generally the case), i.e. represents a field $\Omega \rightarrow \mathrm{P}(2)$ of varying diffusion tensors, the PDE (6) becomes *nonlinear* and can be viewed as the application of temporally and spatially varying *local masks* $G^{\mathbf{T},t}(\mathbf{x})$ over the image $\mathbf{I}$ (Fig.1, bottom row). It particularly shows that *the shapes of each tensor* $\mathbf{T}$ *give the exact smoothing geometry* performed pointwise by the trace-based PDE (6). This local filtering concept makes the link between a generic form of vector-valued diffusion PDE's (6) and *Bilateral filtering* techniques, as described in [2, 23]. A similar approach with non-Gaussian kernels has been also recently proposed for the Beltrami Flow framework [21].

• **Trace-based and Divergence-based tensors** : Differences between divergence tensors $\mathbf{D}$ in (2) and trace tensors $\mathbf{T}$ in (6) can be understood as follows. We develop (2) as :

$$\mathrm{div}\left(\mathbf{D}\nabla I_i\right) = \mathrm{trace}\left(\mathbf{DH}_i\right) + \nabla I_i^T\ \vec{\mathbf{div}}\left(\mathbf{D}\right)$$

where $\vec{\mathbf{div}}\left(\right)$ is defined as a divergence operator acting on matrices and returning vectors :

$$\text{if } \mathbf{D} = (d_{ij}), \qquad \vec{\mathbf{div}}\left(\mathbf{D}\right) = \left(\begin{array}{c} \mathrm{div}\left((d_{11} \quad d_{12})^T\right) \\ \mathrm{div}\left((d_{21} \quad d_{22})^T\right) \end{array}\right)$$

Thus, we see that an additional term $\nabla I_i^T \vec{\mathbf{div}}\left(\mathbf{D}\right)$ appears, connected to the *spatial variation* of the tensor field $\mathbf{D}$. It can perturb the smoothing behavior given by the first part $\mathrm{trace}\left(\mathbf{DH}_i\right)$, which actually corresponds to a local smoothing directed by the spectral elements of $\mathbf{D}$. As a result, the divergence-based equation (2) may smooth the image $\mathbf{I}$ with weights and directions that are different from those given by $\mathbf{D}$. This is actually the case for the ϕ-function formulation (4), where the smoothing process doesn't behave finally (and fortunately) as an isotropic one, despite the isotropic form of the divergence tensor $\mathbf{D} = \phi^{'}\left(\|\nabla I\|\right)/\|\nabla I\|\ \mathbf{Id}$.

• **Developing the divergence form** : If we restrict to the case where the divergence tensor $\mathbf{D}$ depends only on the spectral elements of the structure tensor $\mathbf{G}$, such as :

$$\mathbf{D} = f_1(\lambda_+, \lambda_-)\theta_+\theta_+^T + f_2(\lambda_+, \lambda_-)\theta_-\theta_-^T \qquad (8)$$

with $f_1, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$, (which is the case for existing equations in the literature), then we can develop the corresponding divergence equation $\mathrm{div}\left(\mathbf{D}\nabla I_i\right)$ into a sum of oriented Laplacians, i.e. trace-based PDE's (details in [24, 27]) :

$$\mathrm{div}\left(\mathbf{D}\nabla I_i\right) = \sum_{j=1}^n \mathrm{trace}\left((\delta_{ij}\mathbf{D} + \mathbf{Q}^{ij})\mathbf{H}_j\right) \qquad (9)$$

where the $\mathbf{Q}^{ij}$ designate a *family of* n^2 *matrices* ($i, j = 1..n$), defined as the symmetric parts of the following matrices $\mathbf{P}^{ij}$ (then, $\mathbf{Q}^{ij} = (\mathbf{P}^{ij} + \mathbf{P}^{ij^T})/2$) :

$$\mathbf{P}^{ij} = \alpha\ \nabla I_i^T \nabla I_j \mathbf{Id} + 2\left(\frac{\partial\alpha}{\partial\lambda_+}\theta_+\theta_+^T + \frac{\partial\alpha}{\partial\lambda_-}\theta_-\theta_-^T\right)\nabla I_j \nabla I_i^T \mathbf{G}$$
$$+2\left((\alpha + \frac{\partial\beta}{\partial\lambda_+})\theta_+\theta_+^T + (\alpha + \frac{\partial\beta}{\partial\lambda_-})\theta_-\theta_-^T\right)\nabla I_j \nabla I_i^T$$

with $\alpha = \frac{f_1(\lambda_+,\lambda_-) - f_2(\lambda_+,\lambda_-)}{\lambda_+ - \lambda_-}$, $\beta = \frac{\lambda_+ f_2(\lambda_+,\lambda_-) - \lambda_- f_1(\lambda_+,\lambda_-)}{\lambda_+ - \lambda_-}$

This development (9) expresses a whole range of previously proposed vector-valued regularization algorithms (variational and divergence based PDE's) into an extended trace-based equation, composed of *several diffusion contributions* that have each simple geometric interpretations in terms of local filtering. The interesting point to notice is that *additional diffusion tensors* $\mathbf{Q}^{ij}$ are appearing and contribute to modify the smoothing behavior which is finally *not given by the initial divergence tensor* $\mathbf{D}$.

4. A Unified Expression

From these previous developments, we can now define a *generic vector-valued regularization PDE* :

$$\frac{\partial I_i}{\partial t} = \sum_{j=1}^n \mathrm{trace}\left(\mathbf{A}^{ij}\mathbf{H}_i\right) \qquad (i = 1..n) \quad (10)$$

where the $\mathbf{A}^{ij}$ is a family of 2×2 symmetric matrices, and $\mathbf{H}_i$ designate the Hessian matrix of I_i. It can be equivalently written with a *super-matrix* notation :

$$\frac{\partial \mathbf{I}}{\partial t} = \vec{\mathbf{trace}}\left(\mathcal{AH}\right) \qquad (11)$$

where $\mathcal{A}$ is a *matrix of diffusion tensors* $\mathbf{A}^{ij}$ (and is itself considered as *symmetric*), and $\mathcal{H}$ is a *vector of Hessian matrices* $\mathbf{H}_j$. The matrix product $\mathcal{AH}$ in (11) is then seen *sub-matrix per sub-matrix*, and the operator $\vec{\mathbf{trace}}\left(\right)$ returns the vector in $\mathbb{R}^n$, composed of each sub-matrix trace.

• **Link with existing expressions** : The PDE (10) is a unifying equation that can be used to describe a wide range of vector-valued regularization :
* First, it develops into a very local formulation both variational and divergence-based approaches, that can be written $\frac{\partial I_i}{\partial t} = \mathrm{div}\left(\mathbf{D}\nabla I_i\right)$ as explained in section 2. This particularly includes the papers [5, 11, 12, 16, 18, 20, 22, 26, 28] among others. As described above, the 2×2 tensors $\mathbf{A}^{ij}$ are then defined to be $\mathbf{A}^{ij} = \delta_{ij}\mathbf{D} + \mathbf{Q}^{ij}$. Note that the $\mathbf{Q}^{ij}$ ($i \neq j$) corresponds here to diffusion contributions of other channels I_j in the current one I_i. This *diffusion energy transfer* can be considered as a particular kind of channel *coupling* in the corresponding vector-valued diffusion PDE.
* Second, the PDE (10) also gathers the oriented-Laplacian formulations $\frac{\partial I_i}{\partial t} = \mathrm{trace}\left(\mathbf{TH}_i\right)$, by choosing $A^{ij} = \delta_{ij}\mathbf{T}$. In this case, the matrix $\mathcal{A}$ is diagonal and no diffusion energy transfer occurs between image channels I_i. The vector coupling is only present through the spectral elements $\lambda_\pm$ and $\theta_\pm$ of the structure tensor $\mathbf{G}$. This unifies the formulations proposed for instance in [14, 19, 25, 26].

• **A new regularization PDE** : The generic regularization equation (10) can be specialized, in order to design a new vector-valued regularization PDE that follows desired these local geometric properties :

* We don't want to mix diffusion contributions between image channels. The desired coupling between vector components I_i should only appear in the diffusion PDE through the computation of the structure tensor $\mathbf{G}$. This means that we have to define only one diffusion tensor $\mathbf{A}$, then choose $\mathbf{A}^{ij} = \delta_{ij}\mathbf{A}$. Undesired coupling terms are then avoided.

* On homogeneous regions (corresponding to low vector variations), we want to *smooth isotropically*, in order to remove the noise efficiently with no-preferred directions : $\frac{\partial I_i}{\partial t} \simeq \Delta I_i = \mathrm{trace}\,(\mathbf{H}_i)$. It means that the tensor $\mathbf{A}$ must be *isotropic* in these regions : $\lim_{(\lambda_+ + \lambda_-)\to 0} \mathbf{A} = \alpha\mathbf{Id}$.

* On vector edges (corresponding to high vector variations), we want to perform an *anisotropic smoothing along the vector edges* θ_-, in order to preserve them while removing the noise : $\frac{\partial I_i}{\partial t} = \mathrm{trace}\left(\beta\theta_-{\theta_-}^T\mathbf{H}_i\right)$, where β is a decreasing function, avoiding corners over-smoothing anyway. This corresponds to an *anisotropic* tensor $\mathbf{A}$ in these regions : $\lim_{(\lambda_+ + \lambda_-)\to 0} \mathbf{A} = \beta\theta_-\theta_-^T$.

The following multivalued regularization PDE respects all these local geometric properties :

$$\frac{\partial I_i}{\partial t} = \mathrm{trace}\,(\mathbf{T}\mathbf{H}_i) \qquad (i = 1..n) \qquad (12)$$

where $\mathbf{T}$ is the tensor field defined pointwise as :

$$\mathbf{T} = f_+\left(\sqrt{\lambda_+^* + \lambda_-^*}\right)\ \theta_-^*{\theta_-^*}^T + f_-\left(\sqrt{\lambda_+^* + \lambda_-^*}\right)\ \theta_+^*{\theta_+^*}^T$$

$\lambda_\pm^*$ and $\theta_\pm^*$ are the spectral elements of $\mathbf{G}_\sigma = \mathbf{G} * G_\sigma$, a *gaussian smoothed version of the structure tensor* $\mathbf{G}$, giving a more coherent approximation of the vector variation directions and magnitudes (see [28]). For our experiments in section 6, we chose $f_+(s) = \frac{1}{1+s^2}$ and $f_-(s) = \frac{1}{\sqrt{1+s^2}}$. This is of course one possible choice (inspired from the *hypersurface formulation* of the scalar case [1]) that verifies the above geometric properties, relying on practical experience. The point is that we can freely choose the weighting functions $f_\pm$ to obtain specific regularization behaviors, since we are sure of the local smoothing performed by (12).

5. Numerical schemes

The PDE (12) can be implemented with classical numerical schemes, based on centered spatial discretizations of the gradients and the Hessians [15]. Here we propose an alternative approach based on the local filtering interpretation of trace-based equations (6), (section 3) : the PDE velocity can be locally estimated by applying a spatially varying gaussian smoothing mask $\mathbf{G}^{(\mathbf{T},t)}$ over the image $\mathbf{I}$:

$$\mathrm{trace}\,(\mathbf{T}\mathbf{H}_i) = \sum_{k,l=-1}^{1} \mathbf{G}^{(\mathbf{T},dt)}(k,l)\ I_i(x-k, y-l)$$

Main advantages of this numerical scheme are :
- It preserves the *maximum principle*, since the local filtering is done only with *normalized gaussian kernels*.
- It is more precise, since the computed kernels $\mathbf{G}^{(\mathbf{T},t)}$ do not depend on a discretization in privileged axis x and y. In particular, no (imprecise) second derivatives (in the Hessians H_i) have to be computed (Fig.2).

As for shortcomings of this scheme, we have to mention that it is specially time-consuming, since it needs the computation of several exponentials for each (x, y) and each iteration. For our experiments, we chose 5×5 kernels.

(a) Noisy image (b) Scheme using Hessian discretizations (c) Scheme using local filtering techniques

Figure 2: *Comparisons of numerical schemes.*

6. Applications

We applied our particular regularization PDE (12) to handle these important image-processing issues :

• **Color image restoration** : Fig.3a represents a digital photograph with *real noise*, due to the bad lightning conditions during the snapshot. Our vector-valued regularization PDE (12) can successfully remove the noise, while preserving the important global features of the image.

• **Improvement of lossy compressed images** : Lossy compression algorithms often introduce visible image artefacts : for instance, bloc effects are classical JPEG drawbacks. Using our flow (12) significantly improves the quality of such degraded images (Fig.3b).

• **Color image inpainting** : Image inpainting, recently proposed in [4, 7, 8, 9] consists in filling undesired holes (defined by the user) in images by *interpolating the data* located at the hole's neighborhood. It is possible to do that by applying our regularization PDE (12) only in the holes to fill : boundaries pixels will be diffused until they completely fill the missing regions, in a *structure-preserving way*. We used it to suppress text on images (Fig.3c), remove real objects in photographs (Fig.3d) and reconstruct partially coded images for compression purposes (Fig.3e).

• **Color image magnification** : In the same way, we performed nonlinear image magnification : An image can be magnified by applying our PDE (12) on a linear interpolation of the corresponding small image, excepted on the original known pixels. It suppresses the usual jagging or bloc effects, inherent to classical interpolation methods (Fig.3g).

• **Flow visualization** : Considering a 2D vector field $\mathcal{F} : \Omega \to \mathbb{R}^2$, we have several ways to visualize it. We can first use vectorial graphics, but we have to subsample the field since this kind of representation is not adapted to represent big flows. A better solution is as follows. We smooth a completely noisy image $\mathbf{I}$, with a regularizing flow equivalent to (12) but where $\mathbf{T}$ is directed by the $\mathcal{F}$, instead of the spectral elements of the structure tensor $\mathbf{G}$:

$$\frac{\partial I_i}{\partial t} = \mathrm{trace}\left(\left[\frac{1}{\|\mathcal{F}\|}\mathcal{F}\mathcal{F}^T\right]\mathbf{H}_i\right) \qquad (i=1..n) \qquad (13)$$

Whereas the PDE evolution time t goes by, more global structures of the flow $\mathcal{F}$ appear, i.e. a visualization *scale-space* of $\mathcal{F}$ is constructed. Here, our used regularization equation (13) *ensures that the smoothing of the pixels is done exactly in the direction of the flow* $\mathcal{F}$ (Fig.3f). This is not the case in [3, 6, 10], where the authors based their equations on divergence expressions. Using similar divergence-based techniques would raise a risk of smoothing the image in false directions, as this has been pointed out in section 3.

Conclusion & Perspectives

We proposed a unifying formalism expressing a large set of existing vector-valued regularization approaches within a common framework, adapted to understand the local behavior of regularization PDE's, by explaining the link between diffusion tensors in divergence or trace-based equations and corresponding local gaussian filtering processes. From this study, we defined a specific regularization equation, based on the respect of a coherent anisotropic smoothing preserving the global features of vector images, as well as specific numerical schemes adapted for accurate implementations. The successful application to several image processing issues demonstrated the efficiency of our regularization approach More results can be found in the author's web page : *http://www-sop.inria.fr/odyssee/team/David.Tschumperle*

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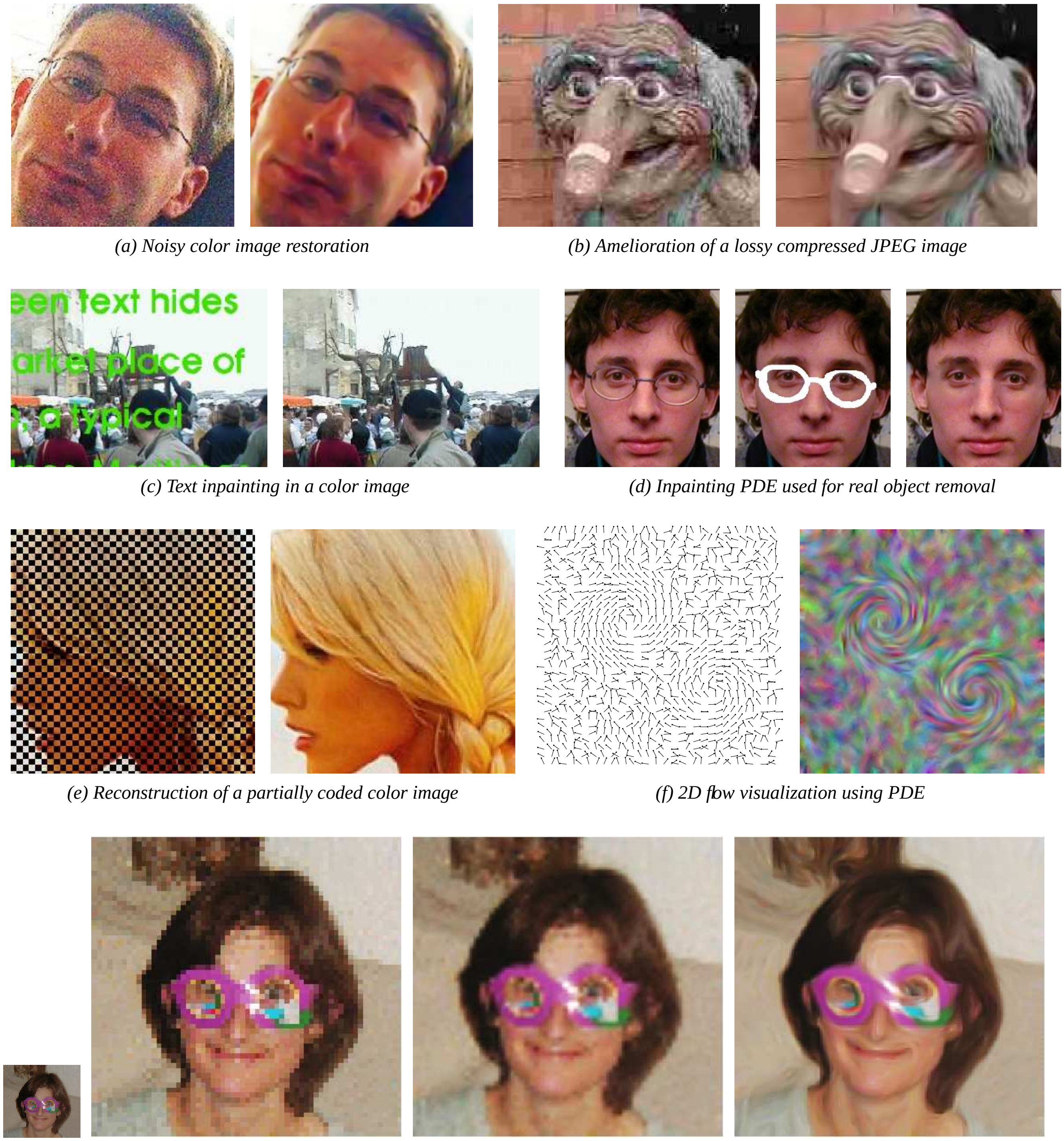

(a) Noisy color image restoration

(b) Amelioration of a lossy compressed JPEG image

(c) Text inpainting in a color image

(d) Inpainting PDE used for real object removal

(e) Reconstruction of a partially coded color image

(f) 2D flow visualization using PDE

(g) Magnification ($\times 4$) of a 64×64 color image, with (from left to right) : bloc magnification, linear interpolation, PDE-based method

Figure 3: **Using our vector-valued regularization PDE's for color image restoration, inpainting, flow visualization and magnification.**