# Homework 2 Solutions for CAP6412

## 1 Question 1-Varying focal length

### 1.1 $rank([\frac{U}{V}])$

Now originally we have that the 2D displacement can be approximated by:

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix}_{2\times 1} = \frac{1}{Z_i}\begin{bmatrix} ft_{X_j} - t_{Z_j}x_i\frac{f}{f_j} \\ ft_{Y_j} - t_{Z_j}y_i\frac{f}{f_j} \end{bmatrix} + \begin{bmatrix} -\frac{\Omega_{X_j}}{f_j}x_iy_i + \Omega_{Y_j}f + \frac{\Omega_{Y_j}}{f_j}x_i^2 - \Omega_{Z_j}y_i + x_i(1-\frac{f}{f_j}) \\ \frac{\Omega_{X_j}}{f_j}x_i^2 + \Omega_{X_j}f + \frac{\Omega_{Y_j}}{f_j}x_iy_i - \Omega_{Z_j}x_i + y_i(1-\frac{f}{f_j}) \end{bmatrix} \tag{1}$$

The displacement $(u_{ij}, v_{ij})$ can be written as

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix}_{2\times 1} = \begin{bmatrix} (M_u)_j \\ (M_v)_j \end{bmatrix}_{2\times 9} P_{i(9\times 1)}$$

where $P_i = \begin{bmatrix} 1 & x_i & y_i & \frac{1}{Z_i} & \frac{x_i}{Z_i} & \frac{y_i}{Z_i} & x_i^2 & y_i^2 & (x_iy_i) \end{bmatrix}^T$
and

$$(M_u)_j = \begin{bmatrix} -f\Omega_{Y_j} & (1-\frac{f}{f_j}) & -\Omega_{Z_j} & ft_{X_j} & -\frac{f}{f_j}t_{Z_j} & 0 & \frac{\Omega_{Y_j}}{f_j} & 0 & -\frac{\Omega_{X_j}}{f_j} \end{bmatrix}$$

$$(M_v)_j = \begin{bmatrix} -f\Omega_{X_j} & \Omega_{Z_j} & (1-\frac{f}{f_j}) & ft_{Y_j} & 0 & -\frac{f}{f_j}t_{Z_j} & 0 & -\frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} \end{bmatrix}$$

Notice that

$$\begin{bmatrix} \frac{U}{V} \end{bmatrix}_{2F\times N} = \begin{bmatrix} \frac{M_U}{M_V} \end{bmatrix}_{2F\times 9} P_{9\times N} \tag{2}$$

and that the $rank([\frac{U}{V}]) \leq 9$.
Now when $t_x = t_y = t_z = 0$ then Equation (2) becomes

$$(M_u)_j = \begin{bmatrix} -f\Omega_{Y_j} & (1-\frac{f}{f_j}) & -\Omega_{Z_j} & 0 & 0 & 0 & \frac{\Omega_{Y_j}}{f_j} & 0 & -\frac{\Omega_{X_j}}{f_j} \end{bmatrix}$$

$$(M_v)_j = \begin{bmatrix} -f\Omega_{X_j} & \Omega_{Z_j} & (1-\frac{f}{f_j}) & 0 & 0 & 0 & 0 & -\frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} \end{bmatrix}$$

Now we observe that the number of zero vectors in **M** is 3, thus these columns of M and corresponding rows of P can be removed $\Rightarrow$ by rank constraint theorem $rank([\frac{U}{V}]) \leq 6$.

### 1.2 $rank([U|V])$

Recall that

$$\begin{bmatrix} u_{ij} & v_{ij} \end{bmatrix}_{1\times 2} = M_{j(1\times 9)} \begin{bmatrix} (P_X)_i \\ (P_Y)_i \end{bmatrix}_{i(9\times 2)}$$

where

$$M_j = \begin{bmatrix} \frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} & f\Omega_{X_j} & \Omega_{Y_j} & \Omega_{Z_j} & ft_{X_j} & ft_{Y_j} & \frac{f}{f_j}t_{Z_j} & (1-\frac{f}{f_j}) \end{bmatrix}$$

and

$$(P_x)_i = \begin{bmatrix} -x_iy_i & x_i^2 & 0 & 1 & -y_i & \frac{1}{Z_i} & 0 & -\frac{x_i}{Z_i} & x_i \end{bmatrix}^T$$

$$(P_y)_i = \begin{bmatrix} -y_i^2 & x_iy_i & -1 & 0 & x_i & 0 & \frac{1}{Z_i} & -\frac{y_i}{Z_i} & y_i \end{bmatrix}^T$$

For all points in all frames

$$\begin{bmatrix} U|V \end{bmatrix}_{(F\times 2N)} = \mathbf{M}_{(F\times 9)} \begin{bmatrix} \mathbf{P_X}|\mathbf{P_Y} \end{bmatrix}_{(9\times 2N)} \tag{3}$$

giving us $rank([U|V]) \leq 9$.

Now when $t_x = t_y = t_z = 0$

$$M_j = \begin{bmatrix} \frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} & f\Omega_{X_j} & \Omega_{Y_j} & \Omega_{Z_j} & 0 & 0 & 0 & (1-\frac{f}{f_j}) \end{bmatrix}$$

the number of zeros in $\mathbf{M}$ is 3, thus these columns of M and corresponding rows of P can be removed $\Rightarrow$ by rank constraint theorem $rank([\frac{U}{V}]) \leq 6$.

## 2 Question 2-Constant Focal Length

### 2.1 $rank([\frac{U}{V}])$

If we expand Equation 2 and substitute $f_i = f$ we directly get

$$\begin{bmatrix} \frac{U}{V} \end{bmatrix}_{2F\times N} = \begin{bmatrix} \frac{M_U}{M_V} \end{bmatrix}_{2F\times 8} P_{8\times N} \tag{4}$$

by a slight rearrangement where

$$P_i = \begin{bmatrix} x_i & y_i & \frac{f}{Z_i} & \frac{x_i}{Z_i} & \frac{y_i}{Z_i} & \frac{(x_iy_i)}{f} & (f+\frac{x_i^2}{f}) & (f+\frac{y_i^2}{f}) \end{bmatrix}^T$$

and

$$(M_u)_j = \begin{bmatrix} 0 & -\Omega_{Z_j} & t_{X_j} & -t_{Z_j} & 0 & -\Omega_{X_j} & \Omega_{Y_j} & 0 \end{bmatrix}$$

$$(M_v)_j = \begin{bmatrix} \Omega_{Z_j} & 0 & t_{Y_j} & 0 & -t_{Z_j} & \Omega_{Y_j} & 0 & -\Omega_{X_j} \end{bmatrix}$$

Now when $t_x = t_y = t_z = 0$ then

$$(M_u)_j = \left[ \begin{array}{cccccccc} 0 & -\Omega_{Z_j} & 0 & 0 & 0 & -\Omega_{X_j} & \Omega_{Y_j} & 0 \end{array} \right]$$

$$(M_v)_j = \left[ \begin{array}{cccccccc} \Omega_{Z_j} & 0 & 0 & 0 & 0 & \Omega_{Y_j} & 0 & -\Omega_{X_j} \end{array} \right]$$

we observe that the number of zero vectors in **M** is 3, thus these columns of M and corresponding rows of P can be removed $\Rightarrow$ by rank constraint theorem $rank([\frac{U}{V}]) \leq 5$.

## 2.2 $rank([U|V])$

$$\left[ \begin{array}{c} U|V \end{array} \right]_{(F\times 2N)} = \mathbf{M}_{(F\times 6)} \left[ \begin{array}{c} \mathbf{P_X}|\mathbf{P_Y} \end{array} \right]_{(6\times 2N)} \tag{5}$$

with

$$M_j = \left[ \begin{array}{cccccc} \Omega_{X_j} & \Omega_{Y_j} & \Omega_{Z_j} & t_{X_j} & t_{Y_j} & t_{Z_j} \end{array} \right]$$

and

$$(P_x)_i = \left[ \begin{array}{cccccc} -\frac{x_i y_i}{f} & (f + \frac{x_i^2}{f}) & -y_i & \frac{f}{Z_i} & 0 & -\frac{x_i}{Z_i} \end{array} \right]^T$$

$$(P_y)_i = \left[ \begin{array}{cccccc} -(f + \frac{y_i^2}{f}) & \frac{x_i y_i}{f} & x_i & 0 & \frac{f}{Z_i} & -\frac{y_i}{Z_i} \end{array} \right]^T$$

Now when $t_x = t_y = t_z = 0$ then

$$M_j = \left[ \begin{array}{cccccc} \Omega_{X_j} & \Omega_{Y_j} & \Omega_{Z_j} & 0 & 0 & 0 \end{array} \right]$$

we can eliminate columns and rows respectively from **M** and **P** to get $rank([U|V]) \leq 3$.