Homework 1 Solutions for CAP6412

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1 Question 1

as we can easily see from figure 1

Derive $\Delta f = 0$ over Ω with $f|\partial\Omega = f^*|\partial\Omega$ from $\min_f \iint_\Omega |\nabla f|^2$ with $f|\partial\Omega = f^*|\partial\Omega$

Solution: Let $F = |\nabla f|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$

according to Euler Lagrange optimization:

$\frac{\partial F}{\partial f} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_x} = 0$, and, $\frac{\partial F}{\partial f} + \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial F}{\partial f_y} = 0$

Notice that $\frac{\partial F}{\partial f} = 0$, $\frac{\mathrm{d}}{\mathrm{d}x} \cdot \frac{\partial F}{\partial f_x} = 2 \cdot (\frac{\partial^2 f}{\partial x^2})$, and, $\frac{\mathrm{d}}{\mathrm{d}y} \cdot \frac{\partial F}{\partial f_y} = 2 \cdot (\frac{\partial^2 f}{\partial y^2})$

$\Rightarrow (\frac{\partial^2 f}{\partial x^2}) = 0$, and,$(\frac{\partial^2 f}{\partial y^2}) = 0$

$\Rightarrow (\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) = 0$ with $f|\partial\Omega = f^*|\partial\Omega$

$\Rightarrow \triangle f = 0$

2 Question 2

Derive $\Delta = \mathrm{div}\mathbf{v}$ over Ω, with $f|\partial\Omega = f^*|\partial\Omega$ from $\min_f \iint_\Omega |\nabla f - \mathbf{v}|^2$ with $f|\partial\Omega = f^*|\partial\Omega$

Figure 1: A nice figure

Again according to Euler Lagrange ...
$\frac{\partial F}{\partial f} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_x} = 0$, and, $\frac{\partial F}{\partial f} + \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial F}{\partial f_y} = 0$
In this case
$\frac{\partial F}{\partial f} = 0$, $\frac{\mathrm{d}}{\mathrm{d}x} \cdot \frac{\partial F}{\partial f_x} = \frac{\mathrm{d}}{\mathrm{d}x} \cdot 2 \cdot (\frac{\partial f}{\partial x} - u) = 2 \cdot (\frac{\partial^2 f}{\partial x^2} - \frac{\partial u}{\partial x}) = 0$, and, $\frac{\mathrm{d}}{\mathrm{d}y} \cdot \frac{\partial F}{\partial f_y} = \frac{\mathrm{d}}{\mathrm{d}y} \cdot 2 \cdot (\frac{\partial f}{\partial y} - v) = 2 \cdot (\frac{\partial^2 f}{\partial y^2} - \frac{\partial v}{\partial y}) = 0$
$\Rightarrow (\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) = (\frac{\partial u}{\partial x}) + (\frac{\partial v}{\partial y})$
$\Rightarrow \Delta f = \mathrm{div}\mathbf{v}$

3 Question 3

Derive $\min_{f|\Omega} \sum_{<p,q>\cap\Omega\neq\emptyset}(f_p - f_q - v_{pq})^2$, with $f_p = f_p^* \; \forall p \in \partial\Omega$

All we do is take the double integral: $\iint_\Omega |\nabla f - \mathbf{v}|^2$ and find the discrete version.

It is a sum over an area $\Rightarrow \nabla f = f_p - f_q$ with $<p,q> \cap\Omega \neq \emptyset$ and $\mathbf{v} = v_{pq}$ a guidance vector.

From here we need to minimize the above equation and separate out the boundary conditions $f_p = f_p^* \; \forall p \in \partial\Omega$

In simple terms this means that $(f_p - f_q - v_{pq}) = 0$ for a neighborhood around point p. To achieve this we notice that $\Rightarrow |N_p| \cdot f_p - \sum_{q\in N_p\cap\Omega} f_q = \sum_{q\in N_p\cap\partial\Omega} f_q^* + \sum_{q\in N_p} v_{pq}$

4 Question 4

How to combine channels similar to how Tschumperle and Deriche did?

Equations $(\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) = 0$ and $(\frac{\partial^2 f}{\partial x^2}) + (\frac{\partial^2 f}{\partial y^2}) = (\frac{\partial u}{\partial x}) + (\frac{\partial v}{\partial y})$ are currently applied to each of the RGB channels separately. All we have to do is add the channels up as in:

$$(\frac{\partial^2 f_r}{\partial x^2}) + (\frac{\partial^2 f_r}{\partial y^2}) + (\frac{\partial^2 f_g}{\partial x^2}) + (\frac{\partial^2 f_g}{\partial y^2}) + (\frac{\partial^2 f_b}{\partial x^2}) + (\frac{\partial^2 f_b}{\partial y^2}) = 0$$

and

$$(\frac{\partial^2 f_r}{\partial x^2})+(\frac{\partial^2 f_r}{\partial y^2})+(\frac{\partial^2 f_g}{\partial x^2})+(\frac{\partial^2 f_g}{\partial y^2})+(\frac{\partial^2 f_b}{\partial x^2})+(\frac{\partial^2 f_b}{\partial y^2}) = (\frac{\partial u_r}{\partial x})+(\frac{\partial v_r}{\partial y})+(\frac{\partial u_g}{\partial x})+(\frac{\partial v_g}{\partial y})+(\frac{\partial u_b}{\partial x})+(\frac{\partial v_b}{\partial y})$$