Multi-Frame Correspondence Estimation Using Subspace Constraints

Michal Irani

Motivation

- Correspondence estimation is a fundamental problem
- Linear subspace constraints (orthographic projection)
  Tomasi, Kanade
- "Aperture problem"
- Correspondence between two frames

Rank of a Matrix

Definition. Consider an M x N matrix $\mathbf{A} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_N \\ \vdots & \vdots & & \vdots \end{pmatrix}$ where each vector v_i has M elements.

Then the rank of $\mathbf{A}$ is the number of linearly independent vectors in the set $\{v_1, \dots, v_N\}$.

We can also represent $\mathbf{A}$ as $\mathbf{A} = \begin{pmatrix} \dots & w_1 & \dots \\ \dots & w_2 & \dots \\ & \vdots & \\ \dots & w_M & \dots \end{pmatrix}$ where each vector w_i has N elements.

Then the **rank** of $\mathbf{A}$ is the number of linearly independent vectors in the set $\{w_1, \dots, w_M\}$.

$\Rightarrow$ for all M x N matrix $\mathbf{A}$, rank($\mathbf{A}$) $\leq$ min(M,N)

Example. Suppose $\mathbf{A}$ is a 6 x 4 matrix. Then rank($\mathbf{A}$) $\leq$ 4.

Subspace constraints on displacement fields

- $I_1, \dots, I_F$ a sequence of F frames
- Arbitrary 3D motions
- I is a reference frame
- (u_{ij}, v_{ij}) displacement of (x_i, y_i) from frame I to frame I_j
- U and V are the matrices of displacements

Displacements U and V

$$U = \begin{bmatrix} u_{11}, & u_{21}, & \dots, & u_{N1} \\ u_{12}, & u_{22}, & \dots, & u_{N2} \\ & & \vdots & \\ u_{1F}, & u_{2F}, & \dots, & u_{NF}, \end{bmatrix}$$

$$V = \begin{bmatrix} v_{11}, & v_{21}, & \dots, & v_{N1} \\ v_{12}, & v_{22}, & \dots, & v_{N2} \\ & & \vdots & \\ v_{1F}, & v_{2F}, & \dots, & v_{NF}, \end{bmatrix}$$

$$\begin{bmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{bmatrix} = R_j \cdot \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} + \vec{t}_j;$$

In the orthographic projection model:

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = \begin{bmatrix} x_{ij} - x_i \\ y_{ij} - y_i \end{bmatrix} = \begin{bmatrix} X_{ij} - X_i \\ Y_{ij} - Y_i \end{bmatrix}$$

$$= \begin{bmatrix} (R_{11_j} - 1) & R_{12_j} & R_{13_j} & t_{X_j} \\ R_{21_j} & (R_{22_j} - 1) & R_{23_j} & t_{Y_j} \end{bmatrix}_{(2\times4)} \times \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}_{(4\times1)}$$

For multiple frames:

$$\begin{bmatrix} U \\ V \end{bmatrix}_{(2F\times N)} = \begin{bmatrix} M_U \\ M_V \end{bmatrix}_{(2F\times4)} \cdot P_{(4\times N)}$$

Rank ≤ 4

For all frame points:

$$\begin{bmatrix} u_{1j}, & u_{2j}, & \dots, & u_{Nj} \\ v_{1j}, & v_{2j}, & \dots, & v_{Nj} \end{bmatrix}_{(2\times N)}$$

$$= \begin{bmatrix} (R_{11_j} - 1) & R_{12_j} & R_{13_j} & t_{X_j} \\ R_{21_j} & (R_{22_j} - 1) & R_{23_j} & t_{Y_j} \end{bmatrix}_{(2\times4)} \times P_{(4\times N)};$$

$$P = \begin{bmatrix} X_1, & X_2, & \dots, & X_N \\ Y_1, & Y_2, & \dots, & Y_N \\ Z_1, & Z_2, & \dots, & Z_N \\ 1 & 1 & \dots, & 1 \end{bmatrix}_{(4\times N)}$$

Rank ≤ 8

$$[U|V]_{(F\times2N)} = [M_U|M_V]_{(F\times8)} \times \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}_{(8\times2N)}$$

Subspace Constraints on Image Brightness

- Compute inter-frame U's and V's
- Project U's and V's into the lower dimensional linear subspace
- It is very noisy
- All U's and V's are treated equally in the subspace projection

Subspace Constraints on Image Brightness

- Point-based constraint (Brightness Constancy constraint)
- Region-based constraint (Lucas and Kanade flow constraint)

Brightness Constancy Constraint

$$I(x_i, y_i) = I_j(x_i + u_{ij}, y_i + v_{ij}); \quad I(x_i - u_{ij}, y_i - v_{ij}) = I_j(x_i, y_i);$$

$$u_{ij} \cdot I_{X_i} + v_{ij} \cdot I_{Y_i} + I_{t_{ij}} = 0; I_{t_{ij}} = I_j(x_i, y_i) - I(x_i, y_i);$$

$$\Delta u_{ij} = u_{ij} - u_{ij}^0; \Delta v_{ij} = v_{ij} - v_{ij}^0;$$

$$I(x_i, y_i) = I_j(x_i + u_{ij}, y_i + v_{ij}) = I_j(x_i + u_{ij}^0 + \Delta u_{ij}, y_i + v_{ij}^0 + \Delta v_{ij});$$

$$I(x_i - \Delta u_{ij}, y_i - \Delta v_{ij}) = I_j(x_i + u_{ij}^0, y_i + v_{ij}^0).$$

$$\Delta u_{ij} I_{x_i} + \Delta v_{ij} I_{y_i} + (I_j(x_i + u_{ij}^0, y_i + v_{ij}^0) - I(x_i, y_i)) = 0;$$

$$u_{ij} \cdot I_{x_i} + v_{ij} \cdot I_{y_i} = -I_{t_{ij}}^0, \quad I_{t_{ij}}^0 = (I_j(x_i + u_{ij}^0, y_i + v_{ij}^0) - I(x_i, y_i) - u_{ij}^0 I_{x_i} - v_{ij}^0 I_{y_i});$$

$$[U|V]_{(F\times 2N)} \cdot \begin{bmatrix} F_x \\ F_y \end{bmatrix}_{(2N\times N)} = F_{T_{(F\times N)}}$$

Brightness Constancy Constraint

$$[U|V]_{(F\times 2N)} \cdot \begin{bmatrix} F_x \\ F_y \end{bmatrix}_{(2N\times N)} = F_{T_{(F\times N)}}$$

$$U = \begin{bmatrix} u_{11}, & u_{21}, & \ldots, & u_{N1} \\ u_{12}, & u_{22}, & \ldots, & u_{N2} \\ & & \vdots & \\ u_{1F}, & u_{2F}, & \ldots, & u_{NF}, \end{bmatrix}$$

$$F_X = \begin{bmatrix} I_{x_1} & 0 & \cdots & 0 \\ 0 & I_{x_2} & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & I_{x_N} \end{bmatrix}$$

$$V = \begin{bmatrix} v_{11}, & v_{21}, & \ldots, & v_{N1} \\ v_{12}, & v_{22}, & \ldots, & v_{N2} \\ & & \vdots & \\ v_{1F}, & v_{2F}, & \ldots, & v_{NF}, \end{bmatrix}$$

$$F_Y = \begin{bmatrix} I_{y_1} & 0 & \cdots & 0 \\ 0 & I_{y_2} & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & I_{y_N} \end{bmatrix}$$

$$F_T = \begin{bmatrix} -I_{t_{11}}^0 & -I_{t_{21}}^0 & \cdots & -I_{t_{N1}}^0 \\ -I_{t_{12}}^0 & -I_{t_{22}}^0 & \cdots & -I_{t_{N2}}^0 \\ & \vdots & & \\ -I_{t_{1F}}^0 & -I_{t_{2F}}^0 & \cdots & -I_{t_{NF}}^0 \end{bmatrix}$$

Brightness Constancy Constraint

- Subspace Constraint on Normal Flow

$$-I_{t_{ij}} = u_{ij} \cdot I_{x_i} + v_{ij} \cdot I_{y_i} = (u_{ij}, v_{ij}) \cdot \nabla I_i$$

- Confidence-Weighted Subspace Projection
  (u_{ij}, v_{ij}) are weighted with (I_{x_i}, I_{y_i})

Lucas and Kanade Constraint

$$E(u_{ij}, v_{ij}) = \sum_{k \in W_i} (u_{ij} \cdot I_{x_k} + v_{ij} \cdot I_{y_k} + I^0_{t_{kj}})^2;$$

$$[u_{ij} v_{ij}] \cdot \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} = [g_{ij} h_{ij}];$$

$$a_i = \sum_k (I_{x_k})^2, b_i = \sum_k (I_{x_k} \cdot I_{y_k})^2, c_i = \sum_k (I_{y_k})^2,$$

$$g_{ij} = \sum_k (I_{x_k} \cdot I^0_{t_{kj}}), h_{ij} = -\sum_k (I_{y_k} \cdot I^0_{t_{kj}})$$

$$[U|V]_{(F \times 2N)} \cdot \begin{bmatrix} A & B \\ B & C \end{bmatrix}_{(2N \times 2N)} = [G|H]_{(F \times 2N)}$$

Lucas and Kanade Constraint

$$[U|V]_{(F\times 2N)} \cdot \begin{bmatrix} A & B \\ B & C \end{bmatrix}_{(2N\times 2N)} = [G|H]_{(F\times 2N)}$$

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & a_N \end{bmatrix}; B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & b_N \end{bmatrix}; \quad C = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ & \vdots & & \\ 0 & 0 & \cdots & c_N \end{bmatrix};$$

$$G = \begin{bmatrix} g_{11} & g_{21} & \cdots & g_{N1} \\ g_{12} & g_{22} & \cdots & g_{N2} \\ & \vdots & & \\ g_{1F} & g_{2F} & \cdots & g_{NF} \end{bmatrix}; H = \begin{bmatrix} h_{11} & h_{21} & \cdots & h_{N1} \\ h_{12} & h_{22} & \cdots & h_{N2} \\ & \vdots & & \\ h_{1F} & h_{2F} & \cdots & h_{NF} \end{bmatrix}$$

Lucas and Kanade Constraint

Covariance-Weighted Subspace Projection

Region-based directional-confidence

Noise Reduction in Image Measurements

- $[U,S,V] =$ SVD of $F_{T_{(F\times N)}}$ or $[G|H]_{(F\times 2N)}$
- $U_{(F\times F)} \cdot S_{(F\times N)or(F\times 2N)} \cdot V^T{}_{(N\times N)or(2N\times 2N)} = \tilde{F}_{T\,(F\times N)}$ or $\left[\tilde{G}\middle|\tilde{H}\right]$

Eliminating the Aperture Problem

$$U = \begin{bmatrix} u_{11}, & u_{21}, & \dots, & u_{N1} \\ u_{12}, & u_{22}, & \dots, & u_{N2} \\ & & \vdots & \\ u_{1F}, & u_{2F}, & \dots, & u_{NF} \end{bmatrix}$$

$$V = \begin{bmatrix} v_{11}, & v_{21}, & \dots, & v_{N1} \\ v_{12}, & v_{22}, & \dots, & v_{N2} \\ & & \vdots & \\ v_{1F}, & v_{2F}, & \dots, & v_{NF} \end{bmatrix}$$

$$\begin{bmatrix} U \\ \hline V \end{bmatrix}_{(2F\times N)} = K_{(2F\times r_2)} \cdot L_{(r_2\times N)}$$

$$\begin{bmatrix} U \\ \hline V \end{bmatrix} = (KM^{-1}) \cdot (ML)$$

$$[U|V] \cdot \begin{bmatrix} A & B \\ B & C \end{bmatrix} = [G|H]$$

Eliminating the Aperture Problem

$$[U_0|V_0]\cdot\begin{bmatrix}A_0 & B_0\\ B_0 & C_0\end{bmatrix}=[G_0|H_0]$$

$$[U_0|V_0]\cdot\begin{bmatrix}A_0 & B_0\\ B_0 & C_0\end{bmatrix}=\left[\tilde{G}_0\middle|\tilde{H}_0\right]$$

$$[U_0|V_0]=\left[\tilde{G}_0\middle|\tilde{H}_0\right]\cdot\begin{bmatrix}A_0 & B_0\\ B_0 & C_0\end{bmatrix}^{-1}$$

$$r_2 = rank\left(\begin{bmatrix}\frac{U_0}{V_0}\end{bmatrix}\cdot\begin{bmatrix}\frac{U_0}{V_0}\end{bmatrix}^T\right)$$

$$K=\begin{bmatrix}\frac{K_U}{K_V}\end{bmatrix}=r_2$$ Eigenvectors

$$U=K_U\cdot L; \qquad V=K_V\cdot L$$

$$[K_U L|K_V L]\cdot\begin{bmatrix}\frac{F_X}{F_Y}\end{bmatrix}=\tilde{F}_T \qquad [K_U L|K_V L]\cdot\begin{bmatrix}A & B\\ B & C\end{bmatrix}=\left[\tilde{G}\middle|\tilde{H}\right]$$

Multi-Point Multi-Frame Algorithm

1. Construct a Gaussian pyramid for all frames
2. For each level in each pyramid level do:
   - (a) Compute A, B, C, G, H
   - (b) Project G and H onto $\tilde{G}$ and $\tilde{H}$
   - (c) Compute reliable displacement estimates only for reliable points, U_0 and V_0
   - (d) Compute basis K from U_0 and V_0
   - (e) Linearly solve for the unknown L using GBC or GLK
   - (f) Compute $\tilde{U}$ and $\tilde{V}$ from K and L
3. Keep iterating to refine $\tilde{U}$ and $\tilde{V}$

Applicability of the Subspace Approach

- Motion approximation
  - Rank depends on the number of frames
  - The number of frames is restricted by the underlying motion model (instantaneous motion model might not be valid)
  - Pure translation (not smooth and not uniform)

Applicability of the Subspace Approach

- Gradient approximation

$$\Delta u_{ij} = u_{ij} - u_{ij}^0; \Delta v_{ij} = v_{ij} - v_{ij}^0;$$

$$I(x_i, y_i) \approx I_j(x_i + u_{ij}, y_i + v_{ij}) = I_j(x_i + u_{ij}^0 + \Delta u_{ij}, y_i + v_{ij}^0 + \Delta v_{ij});$$

$$I(x_i - \Delta u_{ij}, y_i - \Delta v_{ij}) = I_j(x_i + u_{ij}^0, y_i + v_{ij}^0).$$

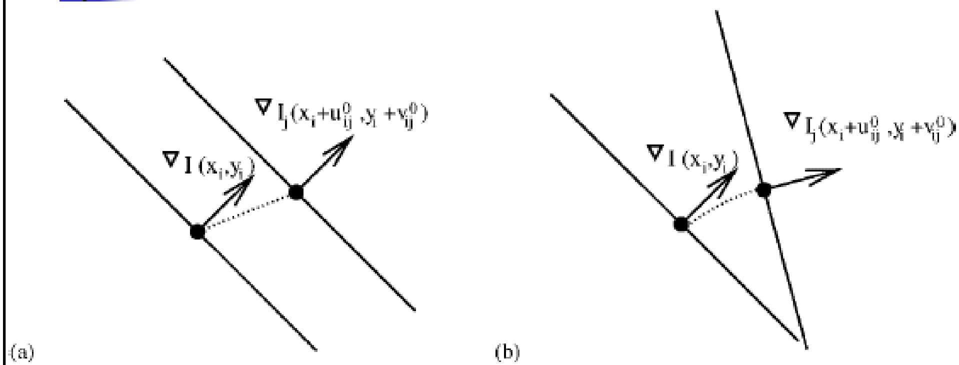
$$[\Delta u_{ij} \Delta v_{ij}] \nabla I_{ij} + I_{t_{ij}} = 0 \qquad [\Delta u_{ij} \Delta v_{ij}] \nabla I_i + I_{t_{ij}} \approx 0$$

$$\nabla I(x_i, y_i) \approx \nabla I_j(x_i + u_{ij}^0, y_i + v_{ij}^0); \nabla I_{ij} \approx \nabla I_i$$

$$\delta\nabla_{ij} = \nabla I_{ij} - \nabla I_i;$$

$$[\Delta u_{ij} \Delta v_{ij}] \delta\nabla_{ij} << [\Delta u_{ij} \Delta v_{ij}] \Delta I_i : \quad \begin{array}{l} i) [\Delta u_{ij} \Delta v_{ij}] \perp \delta\nabla_{ij} \\ ii) \|\delta\nabla_{ij}\| << \|\nabla I_i\|. \end{array}$$

Applicability of the Subspace Approach



Extending the Applicability of Subspace Constraints

- The "Plane + Parallax" Approach

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = -\frac{\gamma_i}{1+\gamma_i \varepsilon_{z_j}} \left(\varepsilon_{z_j} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} \varepsilon_{x_j} \\ \varepsilon_{y_j} \end{bmatrix} \right); where \quad \gamma_i = \frac{H_i}{Z_i}$$

If $\gamma_i \varepsilon_{z_j} << 1$ then rank ≤ 3

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = -\gamma_i \left(\varepsilon_{z_j} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} \varepsilon_{x_j} \\ \varepsilon_{y_j} \end{bmatrix} \right);$$

Ranks for Various World Models, Motion Models, and Camera Models

1. Affine Camera – 3D Scene

$$rank\left(\left[\frac{U}{V}\right]\right) \le 4 \text{ and } rank\left(\left[U \middle| V\right]\right) \le 8$$

2. Affine Camera – Planar (2D) Scene

$$Z_i = \alpha + \beta \cdot X_i + \gamma \cdot Y_i$$

$$rank\left(\left[\frac{U}{V}\right]\right) \le 3 \text{ and } rank\left(\left[U \middle| V\right]\right) \le 6$$

Ranks for Various World Models, Motion Models, and Camera Models

3. Perspective Camera – Instantaneous Motion, 3D Scene

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = \frac{1}{Z_i} \begin{bmatrix} f t_{X_j} - t_{Z_j} x_i \dfrac{f}{f_j} \\ f t_{Y_j} - t_{Z_j} y_i \dfrac{f}{f_j} \end{bmatrix} +$$

$$\begin{bmatrix} -\dfrac{\Omega_{X_j}}{f_j} x_i y_i + \Omega_{Y_j} f + \dfrac{\Omega_{Y_j}}{f_j} x_i^2 - \Omega_{Z_j} y_i + x_i \left(1 - \dfrac{f}{f_j}\right) \\ -\dfrac{\Omega_{X_j}}{f_j} y_i^2 - \Omega_{X_j} f + \dfrac{\Omega_{Y_j}}{f_j} x_i y_i + \Omega_{Z_j} x_i + y_i \left(1 - \dfrac{f}{f_j}\right) \end{bmatrix}$$

Perspective Camera – Instantaneous Motion, 3D Scene

- Varying Focal Length (3D Scene)

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix}_{(2\times1)} = \begin{bmatrix} (M_U)_j \\ (M_V)_j \end{bmatrix}_{(2\times9)} \cdot P_{i(9\times1)} \quad \text{where}$$

$$P_i = \begin{bmatrix} 1 & x_i & y_i & \frac{1}{Z_i} & \frac{x_i}{Z_i} & \frac{y_i}{Z_i} & x_i^2 & y_i^2 & (x_i y_i) \end{bmatrix}^T;$$

$$(M_U)_j = \begin{bmatrix} -f\Omega_{Y_j} & \left(1-\frac{f}{f_j}\right) & -\Omega_{Z_j} & ft_{X_j} & -\frac{f}{f_j}t_{Z_j} & 0 & \frac{\Omega_{Y_j}}{f_j} & 0 & -\frac{\Omega_{X_j}}{f_j} \end{bmatrix};$$

$$(M_V)_j = \begin{bmatrix} -f\Omega_{X_j} & \Omega_{Z_j} & \left(1-\frac{f}{f_j}\right) & ft_{Y_j} & 0 & -\frac{f}{f_j}t_{Z_j} & 0 & -\frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} \end{bmatrix};$$

Perspective Camera – Instantaneous Motion, 3D Scene

$$\begin{bmatrix} U \\ \hline V \end{bmatrix}_{(2F\times N)} = \begin{bmatrix} M_U \\ \hline M_V \end{bmatrix}_{(2F\times9)} \cdot P_{(9\times N)}; \qquad rank\left(\begin{bmatrix} U \\ \hline V \end{bmatrix}\right) \le 9;$$

$$\lfloor u_{ij} \quad v_{ij} \rfloor_{(1\times2)} = M_{j(1\times9)} \left[(P_X)_i \quad (P_Y)_i \right]_{(9\times2)};$$

$$M_j = \begin{bmatrix} \frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} & f\Omega_{X_j} & f\Omega_{Y_j} & \Omega_{Z_j} & ft_{X_j} & ft_{Y_j} & \frac{f}{f_j}t_{Z_j} & \left(1-\frac{f}{f_j}\right) \end{bmatrix};$$

$$(P_X)_i = \begin{bmatrix} -x_i y_i & x_i^2 & 0 & 1 & -y_i & \frac{1}{Z_i} & 0 & -\frac{x_i}{Z_i} & x_i \end{bmatrix}^T;$$

$$(P_Y)_i = \begin{bmatrix} -y_i^2 & x_i y_i & -1 & 0 & x_i & 0 & \frac{1}{Z_i} & -\frac{y_i}{Z_i} & y_i \end{bmatrix}^T;$$

Perspective Camera – Instantaneous Motion, 3D Scene

$$[U|V]_{(F\times 2N)} = M_{(F\times 9)}[P_X|P_Y]_{(9\times 2N)}$$

When both the focal length and the camera motion change across the frames:

$$rank([U|V]) \leq 9 \quad \text{and} \quad rank\left(\begin{bmatrix} U \\ V \end{bmatrix}\right) \leq 9.$$

Perspective Camera – Instantaneous Motion, 3D Scene

- Constant Focal Length (3D Scene)

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = \frac{1}{Z_i}\begin{bmatrix} f t_{X_j} - t_{Z_j} x_i \frac{f}{f_j} \\ f t_{Y_j} - t_{Z_j} y_i \frac{f}{f_j} \end{bmatrix} + \begin{bmatrix} -\frac{\Omega_{X_j}}{f_j} x_i y_i + \left(\Omega_{Y_j} f + \frac{\Omega_{Y_j}}{f_j} x_i^2\right) - \Omega_{Z_j} y_i + x_i\left(1 - \frac{f}{f_j}\right) \\ \left(-\frac{\Omega_{X_j}}{f_j} y_i^2 - \Omega_{X_j} f\right) + \frac{\Omega_{Y_j}}{f_j} x_i y_i + \Omega_{Z_j} x_i + y_i\left(1 - \frac{f}{f_j}\right) \end{bmatrix}$$

$\frac{f}{f_j} \rightarrow 1$

$\left(1 - \frac{f}{f_j}\right) \rightarrow 0$

Perspective Camera – Instantaneous Motion, 3D Scene

$$\begin{bmatrix}u_{ij}\\ v_{ij}\end{bmatrix}=\frac{1}{Z_i}\begin{bmatrix}ft_{X_j}-t_{Z_j}x_i\\ ft_{Y_j}-t_{Z_j}y_i\end{bmatrix}+\begin{bmatrix}-\frac{\Omega_{X_j}}{f}x_iy_i+\Omega_{Y_j}\left(f+\frac{x_i^2}{f}\right)-\Omega_{Z_j}y_i\\ \left(-\frac{y_i^2}{f}-f\right)\Omega_{X_j}+\frac{\Omega_{Y_j}}{f}x_iy_i+\Omega_{Z_j}x_i\end{bmatrix}=$$

$$\begin{bmatrix}0 & -\Omega_{Z_j} & t_{X_j} & -t_{Z_j} & 0 & -\Omega_{X_j} & \Omega_{Y_j} & 0\\ \Omega_{Z_j} & 0 & t_{Y_j} & 0 & -t_{Z_j} & \Omega_{Y_j} & 0 & -\Omega_{X_j}\end{bmatrix}\cdot$$

$$\begin{bmatrix}x_i & y_i & \frac{f}{Z_i} & \frac{x_i}{Z_i} & \frac{y_i}{Z_i} & \frac{x_iy_i}{f} & \left(f+\frac{x_i^2}{f}\right) & \left(f+\frac{y_i^2}{f}\right)\end{bmatrix}^T;$$

$$\begin{bmatrix}U\\ V\end{bmatrix}_{(2F\times N)}=\begin{bmatrix}M_U\\ M_V\end{bmatrix}_{(2F\times 8)}\cdot P_{(8\times N)}.$$

Perspective Camera – Instantaneous Motion, 3D Scene

$$M_j=\begin{bmatrix}\frac{\Omega_{X_j}}{f_j} & \frac{\Omega_{Y_j}}{f_j} & f\Omega_{X_j} & f\Omega_{Y_j} & \Omega_{Z_j} & ft_{X_j} & ft_{Y_j} & \frac{f}{f_j}t_{Z_j} & \left(1-\frac{f}{f_j}\right)\end{bmatrix};$$

$$M_j=\begin{bmatrix}\Omega_{X_j} & \Omega_{Y_j} & \Omega_{Z_j} & t_{X_j} & t_{Y_j} & t_{Z_j}\end{bmatrix};$$

$$(P_X)_i=\begin{bmatrix}-x_iy_i & x_i^2 & 0 & 1 & -y_i & \frac{1}{Z_i} & 0 & -\frac{x_i}{Z_i} & x_i\end{bmatrix}^T;$$

$$(P_X)_i=\begin{bmatrix}-\frac{x_iy_i}{f} & \left(f+\frac{x_i^2}{f}\right) & -y_i & \frac{f}{Z_i} & 0 & -\frac{x_i}{Z_i}\end{bmatrix}^T;$$

$$(P_Y)_i=\begin{bmatrix}-\left(f+\frac{y_i^2}{f}\right) & \frac{x_iy_i}{f} & x_i & 0 & \frac{f}{Z_i} & -\frac{y_i}{Z_i}\end{bmatrix}^T;$$

Perspective Camera – Instantaneous Motion, 3D Scene

When the focal length of the camera remains constant (but unknown) across the sequence, and only the camera motion varies:

$$rank([U|V]) \leq 6 \quad \text{and} \quad rank\left(\begin{bmatrix} U \\ \hline V \end{bmatrix}\right) \leq 8$$

4.Perspective Camera –Instantaneous Motion, Planar (2D) Scene

$$\frac{1}{Z_i} = \alpha' + \beta' \cdot x_i + \gamma' \cdot y_i$$

- Varying Focal Length

$$rank([U|V]) \leq 8 \quad \text{and} \quad rank\left(\begin{bmatrix} U \\ \hline V \end{bmatrix}\right) \leq 6$$

- Constant Focal Length

$$rank([U|V]) \leq 6 \quad \text{and} \quad rank\left(\begin{bmatrix} U \\ \hline V \end{bmatrix}\right) \leq 6$$

Ranks of Planar-Parallax Displacements

$$\begin{bmatrix} u_{ij} \\ v_{ij} \end{bmatrix} = -\gamma_i \left(\varepsilon_{z_j} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} \varepsilon_{x_j} \\ \varepsilon_{y_j} \end{bmatrix} \right) = \begin{bmatrix} \varepsilon_{x_j} & \varepsilon_{z_j} & 0 \\ \varepsilon_{y_j} & 0 & \varepsilon_{z_j} \end{bmatrix} \cdot \begin{bmatrix} \gamma_i & -\gamma_i x_i & -\gamma_i y_i \end{bmatrix}^T;$$

$$\begin{bmatrix} U \\ \hline V \end{bmatrix}_{(2F\times N)} = \begin{bmatrix} M_U \\ \hline M_V \end{bmatrix}_{(2F\times 3)} \cdot P_{(3\times N)};$$

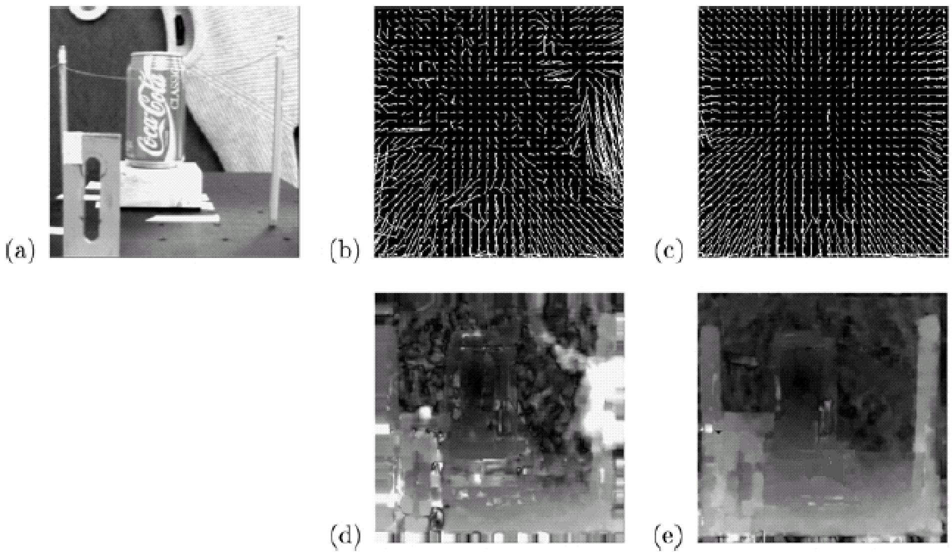
$$\begin{bmatrix} u_{ij} & v_{ij} \end{bmatrix}_{(1\times 2)} = \begin{bmatrix} \varepsilon_{X_j} & \varepsilon_{Y_j} & \varepsilon_{Z_j} \end{bmatrix}_{(1\times 3)} \cdot \begin{bmatrix} \gamma_i & 0 & -\gamma_i x_i \\ 0 & \gamma_i & -\gamma_i y_i \end{bmatrix}^T_{(3\times 2)};$$

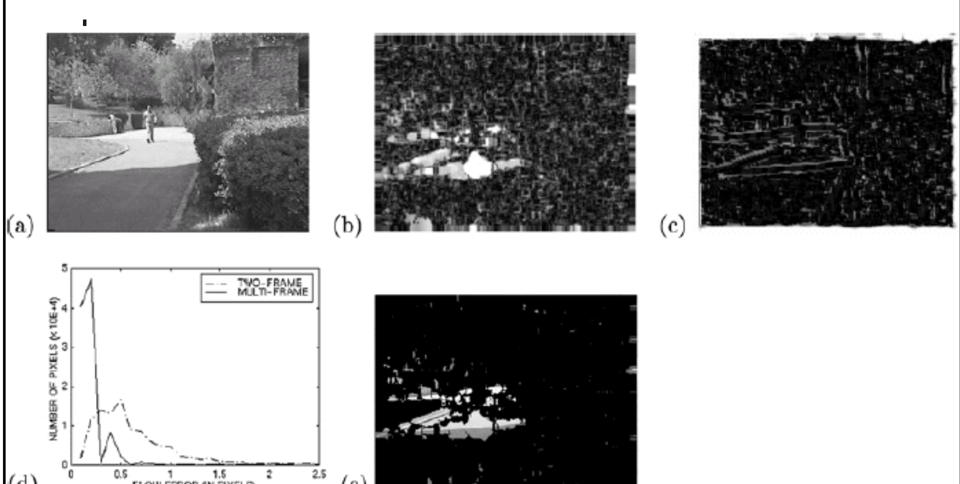
$$\begin{bmatrix} U | V \end{bmatrix}_{(F\times 2N)} = M_{(F\times 3)} \begin{bmatrix} P_X | P_Y \end{bmatrix}_{(3\times 2N)};$$

Ranks of Planar-Parallax Displacements

The planar-parallax displacements reside in 3-dimensional linear subspace, even for extended sequences and for uncalibrated cameras:

$$rank\left(\begin{bmatrix} U | V \end{bmatrix}\right) \le 3 \quad \text{and} \quad rank\left(\begin{bmatrix} U \\ \hline V \end{bmatrix}\right) \le 3$$


(a)
(b)
(c)
(d)
(e)


(a)
(b)
(c)
(d)
(e)
TWO-FRAME
MULTI-FRAME
NUMBER OF PIXELS (x 10E+4)
FLOW ERROR (IN PIXELS)