Levenberg-Marquardt

Optimization

- People optimize
  - Stocks
  - Job
  - exam
- Convert qualitative description into quantitative function
  - Objective function
  - Variables
  - constraints

Examples

- Transportation problem
- Chess playing
- Robot path planning
- Computing the optimal shape of an automobile or aircraft
- Controlling a chemical process or a mechanical device to optimize or meet standards of robustness
- Computer Vision
  - Camera Pose estimation
  - Optical flow
  - Stereo depth estimation
  - etc

Optimization

- Minima, maxima or zero of a function
- Local minima vs global minima

Optimization Problems

- Single variable
- Multiple variables
- Linear
- Non-linear
- Unconstraint optimization
- Constraint optimization

$$\min (x_1 - 2)^2 + (x_2 - 1)^2$$

$$\text{subject to} \left\{ x_1^2 - x_2 \le 0, x_1 + x_2 \le 2 \right.$$

Desirable Properties

- Robustness
- Accuracy
- Efficiency

Iterative Solution

$$X^0, X^1, X^2, \ldots X^n$$

- Initial estimate

$$X^n \approx X^{n-1}$$

$$X^n \approx P$$

- Convergence
  - Linear
  - Super linear
  - Quadratic

Rate of Convergence

Definition : Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p and that $e_n = p_n - p$

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty}\frac{|e_{n+1}|}{|e_n|^{\alpha}} = \lambda$$

then the seq is said to converge to p of order α with asymptotic error constant λ.

$\alpha = 1$, linear

$\alpha = 2$, quadratic

$\alpha = 1$, and $\lambda = 0$, superlinear

Numerical Optimization

- Computation of
  - derivatives,
  - gradient,
  - Jacobian,
  - Hessian
- Analytical derivatives not possible
- Numerical derivatives, finite difference
  - Solution of a linear system (Inverse of a matrix)

$$\text{Derivative: } f'(x) = \frac{df}{dx}, \ x \text{ is a scalar}$$

$$\text{Gradient: } \nabla f(x_1, x_2, \ldots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Jacobian

$$F(x_1, x_2, \ldots, x_n) = f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)$$

$$J(F) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Hessian

$$f(x_1, x_2, \ldots, x_n)$$

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Optimization Methods

- Gradient Descent
- Conjugate Gradient
- Newton
- Quasi Newton
- Levenberg Marquadet

Weighted Non-linear least squares fit

Consider a set of non-linear equations:

$$y_i = f(x_i, a)$$

Our aim is to determine *a* vector such that the following is minimized:

$$\Psi(a) = \sum_{i=1}^{N} \left(\frac{y_i - f(x_i, a)}{\sigma_i} \right)^2$$

weights

Function to be minimized $$\Psi(a)=\sum_{i=1}^{N}\left(\frac{y_i - f(x_i,a)}{\sigma_i}\right)^2 \qquad \text{(D)}$$

Newton's (Inverse Hessian) method:

$$a_{next} = a_{current} + D^{-1}[-\nabla\Psi(a_{current})] \qquad \text{(A)}$$

Where D is a Hessian matrix

The gradient is given by:

$$\frac{\partial\Psi}{\partial a_k} = -2\sum_i\left[\frac{y_i - f(x_i,a)}{\sigma_i^2}\right]\frac{\partial f(x_i,a)}{\partial a_k} \quad k=1,\ldots,m$$

The Hessian is given by:

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,y)}{\partial a_k}\frac{\partial f(x_i,y)}{\partial a_l} - [y_i - f(x_i,a)]\frac{\partial^2 f(x_i,a)}{\partial a_k \partial a_l}\right] \qquad \text{(B)}$$

Let us define:

$$\beta_k \equiv -\frac{1}{2}\frac{\partial \Psi}{\partial \mathrm{a}_\mathrm{k}} \qquad \alpha_{kl} \equiv \frac{1}{2}\frac{\partial^2 \Psi}{\partial \mathrm{a}_\mathrm{k} \partial a_l}$$

Now the Hessian is given by:

$$[\alpha] = \frac{1}{2} D$$

Newton's method (A) $\quad a_{next} = a_{current} + D^{-1}[-\nabla \Psi(a_{current})] \quad$ (A)

can be written as

$$\sum_{l=1}^{M} \alpha_{kl} \delta a_l = \beta_k \qquad \text{(E)}$$

Assume $\delta a = a_{next} - a_{current}$

The gradient descent is given by:

$$a_{next} = a_{current} + const[-\nabla\Psi(a_{current})] \qquad \text{(C)}$$

$$\beta_k \equiv -\frac{1}{2}\frac{\partial\Psi}{\partial a_k}$$

$$\delta a_l = const\ \beta_l \qquad \delta a = a_{next} - a_{current}$$

Assume the second term in (B) is zero:

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l} - [y_i - f(x_i,a)]\frac{\partial^2 f(x_i,a)}{\partial a_k \partial a_l}\right] \qquad \text{(B)}$$

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l}\right]$$

Now

$$\alpha_{kl} = \sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l}\right]$$

$$\beta_k \equiv -\frac{1}{2}\frac{\partial \Psi}{\partial \mathrm{a}_\mathrm{k}} \qquad \alpha_{kl} = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i, a)}{\partial a_k}\frac{\partial f(x_i, a)}{\partial a_l}\right]$$

$$\delta a_l = const\ \beta_l \qquad \text{(C)}$$

Gradient descent

What should be the constant?

The units of β_k are $1/a_k$,
there fore units of constant
should be a_k^2

Only component of with this property is : $1/\alpha_{kk}$

$$\delta a_l = const\ \beta_l \qquad \text{(C)}$$

Gradient descent

Let the constant be given by

$$const = \frac{1}{\lambda \alpha_{ll}} \qquad \delta a_l = \frac{1}{\lambda \alpha_{ll}} \beta_l \qquad \lambda \alpha_{ll} \delta a_l = \beta_l \qquad \text{(G)}$$

Newton from (E)

$$\sum_{l=1}^{M} \alpha_{kl} \delta a_l = \beta_k$$

Now define:

$$\alpha'_{jj} \equiv \alpha_{jj}(1+\lambda) \quad \text{for } i = j$$

$$\alpha'_{kj} \equiv \alpha_{kj} \quad \text{when j} \neq \text{k} \qquad \text{(F)}$$

Combining (E) and (G) and using (F)

$$\sum_{l=1}^{m} \alpha'_{kl} \delta a_l = \beta_k \qquad \text{(H)}$$

L-M

Algorithm

1. Start with some initial estimate of *a*.

2. Compute $\Psi(x_i, a)$ (equation D). $\Psi(a) = \sum_{i=1}^{N}\left(\frac{y_i - f(x_i, a)}{\sigma_i}\right)^2$

3. Pick a modest value of λ=.001.

4. Solve linear system (H) for δa and evaluate $\Psi(x_i, a+\delta a)$ $\sum_{l=1}^{m} \alpha'_{kl}\delta a_l = \beta_k$

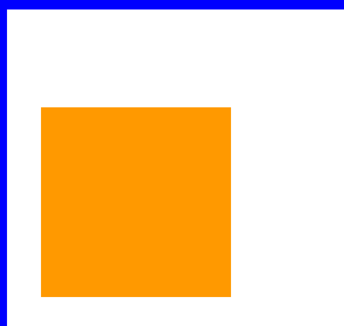
5. If $\Psi(x_i, a+\delta a) \geq \Psi(x_i, a)$, increase λ by a factor of 10, and go to step (4)

6. If $\Psi(x_i, a+\delta a) \leq \Psi(x_i, a)$ decrease λ by a factor of 10, update the trial solution: $a \leftarrow a+\delta a$, and go back to step 4.

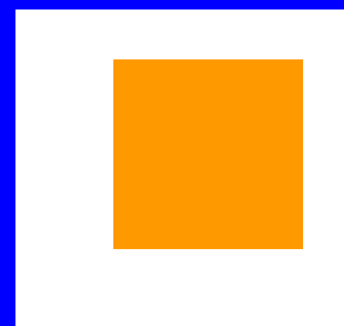
Szeliski

Projective

Projective (Homogpraphy)



$f(X', t-1)$



$f(X, t)$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

Feature-based Estimation of Homorgrapy

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_1}{c_1 x + c_2 y + 1}$$

$$\begin{bmatrix} x'_k \\ y'_k \end{bmatrix} = \begin{bmatrix} x_k & y_k & 1 & 0 & 0 & 0 & -x_k x'_k & -y_k x'_k \\ 0 & 0 & 0 & x_k & y_k & 1 & -x_k y'_k & -y_k y'_k \end{bmatrix} \mathbf{a}$$

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & b_1 & a_3 & a_4 & b_2 & c_1 & c_1 \end{bmatrix}^T$$

Feature-based Estimation of Homorgrapy

$$\begin{bmatrix} x_1' \\ y_1' \\ \\ \\ x_k' \\ y_k' \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x_1' & -y_1x_1' \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y_1' & -y_1y_1' \\ \\ \\ x_k & y_k & 1 & 0 & 0 & 0 & -x_kx' & -y_kx_k' \\ 0 & 0 & 0 & x_k & y_k & 1 & -x_ky_k' & -y_ky' \end{bmatrix} \mathbf{a}$$

$$\mathbf{P} = \mathbf{Aa}$$

Perform least squares fit to compute a.

Szeliski (Image-based estimation of Homography)

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

Projective

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$E = \sum \left[f(x', y') - f(x, y) \right]^2 = \sum e^2$$

min

Szeliski

Motion Vector:

$$\mathbf{m} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & c_1 & c_2 \end{bmatrix}^T$$

Szeliski (Levenberg-Marquadet)

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$\alpha_{kl} = \sum_n \frac{\partial e_n}{\partial m_k} \frac{\partial e_n}{\partial m_l}$$

$$\beta_k = -\sum e \frac{\partial e_n}{\partial m_k}$$ **gradient**

$$\Delta m = (A + \lambda I)^{-1} b$$

Approximation of Hessian (*$J^T J$, Jacobian)*

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'} \frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'} \frac{\partial y'}{\partial a_1}$$

Approximation of Hessian

$$J^T = \begin{bmatrix} \frac{\partial e_1}{\partial m_1} & \frac{\partial e_2}{\partial m_1} & \cdots & \frac{\partial e_n}{\partial m_1} \\ \frac{\partial e_1}{\partial m_2} & \frac{\partial e_2}{\partial m_2} & \cdots & \frac{\partial e_n}{\partial m_2} \\ \frac{\partial e_1}{\partial m_3} & \frac{\partial e_2}{\partial m_3} & \cdots & \frac{\partial e_n}{\partial m_3} \\ \frac{\partial e_1}{\partial m_4} & \frac{\partial e_2}{\partial m_4} & \cdots & \frac{\partial e_n}{\partial m_4} \\ \frac{\partial e_1}{\partial m_5} & \frac{\partial e_2}{\partial m_5} & \cdots & \frac{\partial e_n}{\partial m_5} \\ \frac{\partial e_1}{\partial m_6} & \frac{\partial e_2}{\partial m_6} & \cdots & \frac{\partial e_n}{\partial m_6} \\ \frac{\partial e_1}{\partial m_7} & \frac{\partial e_2}{\partial m_7} & \cdots & \frac{\partial e_n}{\partial m_7} \\ \frac{\partial e_1}{\partial m_8} & \frac{\partial e_2}{\partial m_8} & \cdots & \frac{\partial e_n}{\partial m_8} \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial m_1} & \frac{\partial e_1}{\partial m_2} & \frac{\partial e_1}{\partial m_3} & \frac{\partial e_1}{\partial m_4} & \frac{\partial e_1}{\partial m_5} & \frac{\partial e_1}{\partial m_6} & \frac{\partial e_1}{\partial m_7} & \frac{\partial e_1}{\partial m_8} \\ \frac{\partial e_2}{\partial m_1} & \frac{\partial e_2}{\partial m_2} & \frac{\partial e_2}{\partial m_3} & \frac{\partial e_2}{\partial m_4} & \frac{\partial e_2}{\partial m_5} & \frac{\partial e_2}{\partial m_6} & \frac{\partial e_2}{\partial m_7} & \frac{\partial e_2}{\partial m_8} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial e_n}{\partial m_1} & \frac{\partial e_n}{\partial m_2} & \frac{\partial e_n}{\partial m_3} & \frac{\partial e_n}{\partial m_4} & \frac{\partial e_n}{\partial m_5} & \frac{\partial e_n}{\partial m_6} & \frac{\partial e_n}{\partial m_7} & \frac{\partial e_n}{\partial m_8} \end{bmatrix}$$

$$A = J^T J$$

A Matrix

$$\alpha_{kl} = \sum_n \frac{\partial e_n}{\partial m_k} \frac{\partial e_n}{\partial m_l}$$

Gradient Vector

$$b = \begin{bmatrix} -\sum_n e_n \frac{\partial e_n}{\partial a_1} \\ -\sum_n e_n \frac{\partial e_n}{\partial a_2} \\ -\sum_n e_n \frac{\partial e_n}{\partial a_3} \\ -\sum_n e_n \frac{\partial e_n}{\partial a_4} \\ -\sum_n e \frac{\partial e_n}{\partial b_1} \\ -\sum_n e_n \frac{\partial e_n}{\partial b_2} \\ -\sum_n e_n \frac{\partial e_n}{\partial c_1} \\ -\sum_n e_n \frac{\partial e_n}{\partial c_2} \end{bmatrix}$$

$$b = \begin{bmatrix} -\sum_n e_n \frac{\partial e_n}{\partial m_1} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_2} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_3} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_4} \\ -\sum_n e \frac{\partial e_n}{\partial m_5} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_6} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_7} \\ -\sum_n e_n \frac{\partial e_n}{\partial m_8} \end{bmatrix}$$

Partial Derivatives wrt motion parameters

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_1}$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}, \quad y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_1} = \frac{x}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_1} = 0$$

$$\frac{\partial x'}{\partial a_2} = \frac{y}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_2} = 0$$

$$\frac{\partial x'}{\partial a_3} = 0 \qquad \frac{\partial y'}{\partial a_3} = \frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_4} = 0 \qquad \frac{\partial y'}{\partial a_4} = \frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial b_1} = \frac{1}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial b_1} = 0$$

$$\frac{\partial x'}{\partial b_2} = 0 \qquad \frac{\partial y'}{\partial b_2} = \frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial c_1} = \frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_1} = \frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial x'}{\partial c_2} = \frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_2} = \frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

Partial derivatives wrt image coordinates

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$\frac{\partial e}{\partial x'} = f_{x'}$$

$$\frac{\partial e}{\partial y'} = f_{y'}$$

Partial derivatives

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_1} = f_{x'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_2} = f_{x'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_3} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_3} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_3} = f_{y'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_4} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_4} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_4} = f_{y'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_1} = f_{x'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_2} = f_{y'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial c_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_1} = f_{x'}\frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial e}{\partial c_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_2} = f_{x'}\frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

Gradient Vector

$$\mathbf{b} = \begin{bmatrix} -\sum e f_{x'} \dfrac{x}{c_1 x + c_2 y + 1} \\ -\sum e f_{x'} \dfrac{y}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'} \dfrac{x}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'} \dfrac{y}{c_1 x + c_2 y + 1} \\ -\sum e f_{x'} \dfrac{1}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'} \dfrac{1}{c_1 x + c_2 y + 1} \\ \sum e x \left[\dfrac{f_{x'}(a_1 x + a_2 y + b_1) + f_{y'}(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2} \right] \\ \sum e y \left[\dfrac{f_{x'}(a_1 x + a_2 y + b_1) + f_{y'}(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2} \right] \end{bmatrix}$$

Szeliski (Levenberg-Marquadet)

- Start with some initial value of m, and λ=.001
  - For each pixel I at (x_i, y_i)
  - Compute (x', y') using projective transform.
  - Compute $e = f(x', y') - f(x, y)$
  - Compute $\frac{\partial e}{\partial m_k} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial m_k} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial m_k}$

Szeliski (Levenberg-Marquadet)

-Compute A **and** b

-Solve system

$$(A - \lambda I)\Delta m = b$$

-Update

$$m^{t+1} = m^{t} + \Delta m$$

Szeliski (Levenberg-Marquadet)

- check if error has decreased, if not increase λ by a factor of 10 and compute a new Δm
- If error has decreased, decrease λ by a factor of 10 and compute a new Δm
- Continue iteration until error is below threshold.