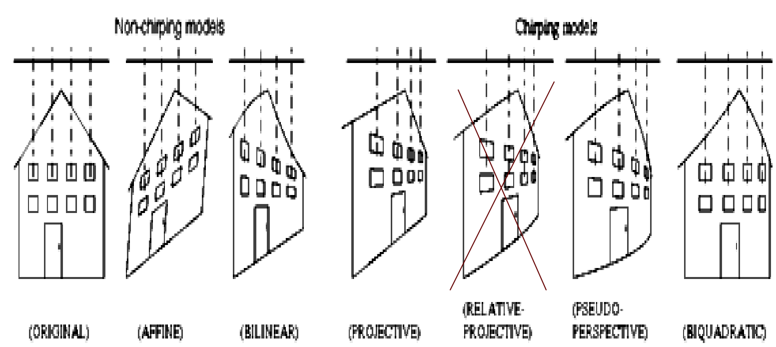
Lecture 3

Displacement Models (contd)



Affine Mosaic

Projective Mosaic

Instantaneous Velocity Model

3-D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} X'-X \\ Y'-Y \\ Z'-Z \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \left(\begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

3-D Rigid Motion

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$\dot{\mathbf{X}} = \Omega \times \mathbf{X} + \mathbf{V}$$

Cross Product

Orthographic Projection

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$y = Y$$
$$x = X$$

$$u = \dot{x} = \Omega_2 Z - \Omega_3 y + V_1$$
$$v = \dot{y} = \Omega_3 x - \Omega_1 Z + V_2$$

(u,v) is optical flow

Plane+orthographic(Affine)

$$Z = a + bX + cY$$

$$u = V_1 + \Omega_2 Z - \Omega_3 y$$
$$v = V_2 + \Omega_3 x - \Omega_1 Z$$

(1) Show this

$$u = b_1 + a_1 x + a_2 y$$
$$v = b_2 + a_3 x + a_4 y$$

$$\Downarrow$$

$$\mathbf{u} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

$$b_1 = V_1 + a\Omega_2$$
$$a_1 = b\Omega_2$$
$$a_2 = c\Omega_2 - \Omega_3$$

$$b_2 = V_2 - a\Omega_1$$
$$a_3 = \Omega_3 - b\Omega_1$$
$$a_4 = -c\Omega_1$$

Perspective Projection (arbitrary flow)

$$x = \frac{fX}{Z} \qquad u = \dot{x} = \frac{fZ\dot{X} - fX\dot{Z}}{Z^2} = f\frac{\dot{X}}{Z} - x\frac{\dot{Z}}{Z}$$

$$y = \frac{fY}{Z} \qquad v = \dot{y} = \frac{fZ\dot{Y} - fY\dot{Z}}{Z^2} = f\frac{\dot{Y}}{Z} - y\frac{\dot{Z}}{Z}$$

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$u = f\left(\frac{V_1}{Z} + \Omega_2\right) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$

$$v = f\left(\frac{V_2}{Z} - \Omega_1\right) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

Plane+Perspective (pseudo perspective)

$$u = f\left(\frac{V_1}{Z} + \Omega_2\right) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$

$$v = f\left(\frac{V_2}{Z} - \Omega_1\right) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

$$Z = a + bX + cY$$

$$\frac{1}{Z} = \frac{1}{a} - \frac{b}{a}x - \frac{c}{a}y$$

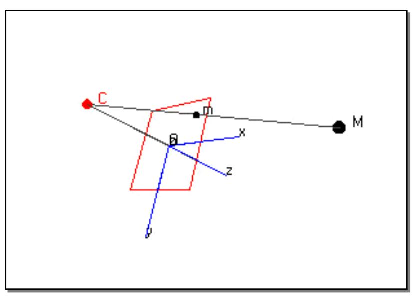
(2) Show this

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$$

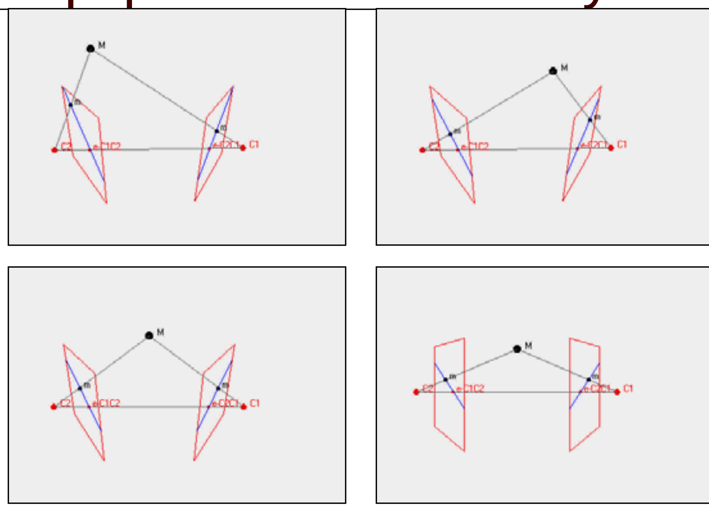
$$v = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2$$

3D Scenes

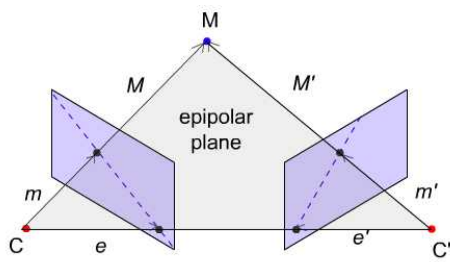
- When the scene is neither planar not distant, the relation between two consecutive frames is described by the *fundamental matrix*.
- What the heck is epipolar geometry?



Epipolar Geometry



Epipolar Geometry



- *Coplanar Condition*

  $(\boldsymbol{B} - \boldsymbol{A}) \cdot \boldsymbol{A} \times \boldsymbol{B} = 0$

- $\boldsymbol{M}' = \boldsymbol{R}\,(\boldsymbol{M} - \boldsymbol{T})$
- $(\boldsymbol{M} - \boldsymbol{T})^{T} . \boldsymbol{T} \times \boldsymbol{M} = 0$
- $\boldsymbol{R}^{-1} = \boldsymbol{R}^{T}$
- $(\boldsymbol{R}^{T} \boldsymbol{M}')^{T} . \boldsymbol{T} \times \boldsymbol{M} = 0$
- $\boldsymbol{A} \times \boldsymbol{B} = S \cdot \boldsymbol{B}$ *therefore* $\boldsymbol{M}'^{T} \boldsymbol{R}\, S\, \boldsymbol{M} = 0$ a1
- $\boldsymbol{M}'^{T} \boldsymbol{E}\, \boldsymbol{M} = 0$ *where* $\boldsymbol{E} = \boldsymbol{RS}$ *is the essential matrix*

Slide 14

a1 The Essential Matrix is the mapping between points and epipolar lines
subail, 10/9/2001

Vector Cross Product to Matrix-vector multiplication

$$A \times B = A.S$$

$$S = \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix}$$

Camera Model Revisited: Perspective

$$C_h = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$x = \frac{fX}{Z}$$

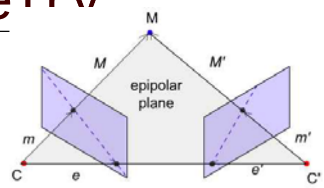
$$y = \frac{fY}{Z}$$

Origin at the lens
Image plane in front of the lens

$$m = PM$$

$$P^{-1}m = M$$

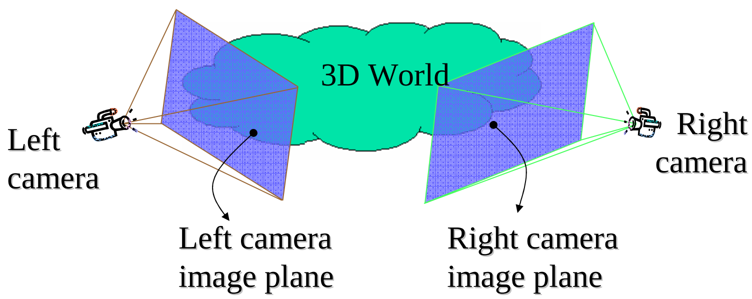
Epipolar Geometry



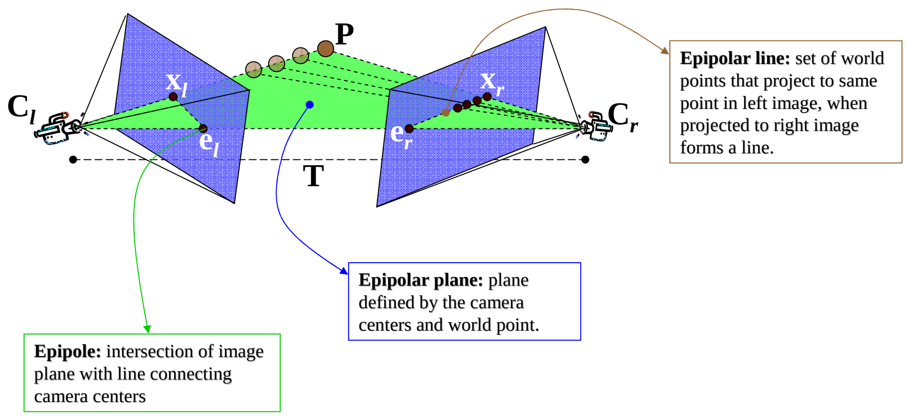
- The *Fundamental Matrix* is the *Essential Matrix* described in pixel coordinates.
- m and m′ are the points in pixel coordinates corresponding to **m** and **m**′
- $\boldsymbol{M} = \boldsymbol{P}^{-1}$ m and $\boldsymbol{M}' = \boldsymbol{P}'^{-1}$ m′
- $\boldsymbol{m} = f/Z \,.\, \boldsymbol{M}$ and $\boldsymbol{m}' = f'/Z' \, \boldsymbol{M}'$
- $\boldsymbol{M}'^{T} \boldsymbol{E} \boldsymbol{M} = 0$
- $\boldsymbol{m}'^{T} \boldsymbol{F} \boldsymbol{m} = 0$
- $(\boldsymbol{P}'^{-1} m')^{T} \boldsymbol{E} (\boldsymbol{P}^{-1} m) = m'^{T} \boldsymbol{P}'^{-T} \boldsymbol{E} \boldsymbol{P}^{-1} m$

Epipolar Geometry

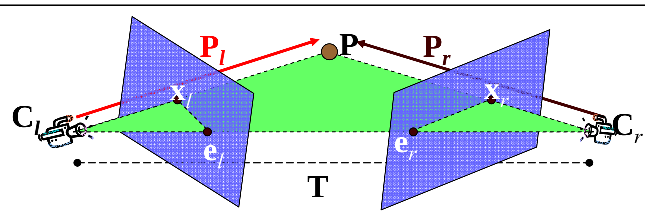
- Defined for two static cameras



Epipolar Geometry



Essential Matrix



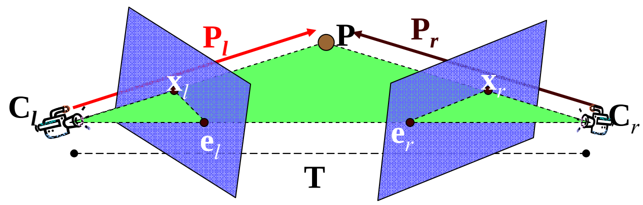
Coplanarity constraint between vectors ($\mathbf{P}_l$-**T**), **T**, $\mathbf{P}_l$.

$$\left.\begin{array}{l}(\mathbf{P}_l - \mathbf{T})^{\mathrm{T}}\,\mathbf{T}\times\mathbf{P}_l = 0 \\ \mathbf{P}_r = \mathbf{R}(\mathbf{P}_l - \mathbf{T})\end{array}\right\} \quad \mathbf{P}_r^{\mathrm{T}}\mathbf{R}\,\mathbf{S}\,\mathbf{P}_l = 0 \quad \Rightarrow \quad \mathbf{P}_r^{\mathrm{T}} E \mathbf{P}_l = 0$$

essential matrix

$$\mathbf{S} = \begin{pmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{pmatrix}$$

Fundamental Matrix



relative rotation

$$\mathbf{P}_r = \mathbf{R}(\mathbf{P}_l - \mathbf{T})$$
$$\mathbf{P}_l = R_l\mathbf{P} + T_l$$
$$\mathbf{P}_r = R_r\mathbf{P} + T_r$$

$$\mathbf{R} = R_r R_l^{\mathrm{T}} \quad \text{(A)}$$
$$\mathbf{T} = T_l - \mathbf{R}^{\mathrm{T}} T_r \quad \text{(B)}$$

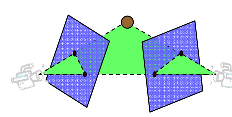
relative translation

$$\mathbf{x}_l = M_l\mathbf{P}_l$$
$$\mathbf{x}_r = M_r\mathbf{P}_r$$
$$\mathbf{P}_r^{\mathrm{T}} E\mathbf{P}_l = 0$$

$$\mathbf{x}_r^{\mathrm{T}}\left(M_r^{-\mathrm{T}} E M_l^{-1}\right)\mathbf{x}_l = 0$$
$$\boxed{\mathbf{x}_r^{\mathrm{T}} F\mathbf{x}_l = 0}$$

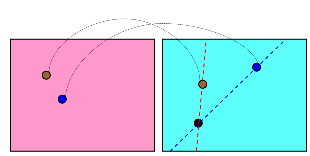
fundamental matrix

Fundamental Matrix

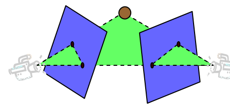


$$\mathbf{x}'^{\mathrm{T}} F\mathbf{x} = \mathbf{x}'^{\mathrm{T}}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix}\mathbf{x} = 0$$

- Rank 2 matrix (due to **S**)
- Given a point in left camera **x**, epipolar line in right camera is: $\mathbf{u}_r = F\mathbf{x}$



Fundamental Matrix



$$\mathbf{x}_l{}^{\mathrm{T}} F \mathbf{x}_r = \mathbf{x}_l{}^{\mathrm{T}} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix} \mathbf{x}_r = 0$$

- 3x3 matrix with 9 components
- Rank 2 matrix (due to S)
- 7 degrees of freedom
- Given a point in left camera $\mathbf{x}$, epipolar line in right camera is: $\mathbf{u}_r = F\mathbf{x}$

Fundamental Matrix

- Longuet Higgins (1981)
- Hartley (1992)
- Faugeras (1992)
- Zhang (1995)

Fundamental Matrix

- Fundamental matrix captures the relationship between the corresponding points in two views.

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11}x' + f_{12}y'_i + f_{13} \\ f_{21}x' + f_{22}y'_i + f_{23} \\ f_{31}x' + f_{32}y'_i + f_{33} \end{bmatrix} = 0,$$

$$x_i(f_{11}x' + f_{12}y'_i + f_{13}) + y_i(f_{21}x' + f_{22}y'_i + f_{23}) + (f_{31}x' + f_{32}y'_i + f_{33}) = 0$$

Fundamental Matrix

$$x_i(f_{11}x' + f_{12}y'_i + f_{13}) + y_i(f_{21}x' + f_{22}y'_i + f_{23}) + (f_{31}x' + f_{32}y'_i + f_{33}) = 0$$

$$x_i x' f_{11} + x_i y'_i f_{12} + x_i f_{13} + y_i x' f_{21} + x' y'_i f_{22} + y'_i f_{23} + x' f_{31} + y'_i f_{32} + f_{33} = 0$$

$$Mf = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ x'_2x_2 & x'_2y_2 & x'_2 & y'_2x_2 & y'_2y_2 & y'_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_1 & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} f = 0$$

M is 9 by n matrix $\quad \boldsymbol{f} = [\boldsymbol{f_{11}} \; \boldsymbol{f_{12}} \; \boldsymbol{f_{13}} \; \boldsymbol{f_{21}} \; \boldsymbol{f_{22}} \; \boldsymbol{f_{23}} \; \boldsymbol{f_{31}} \; \boldsymbol{f_{32}} \; \boldsymbol{f_{33}}]$

To solve the equation, the rank(M) must be 8.

Computation of Fundamental Matrix

Normalized 8-point algorithm

Objective:

Compute fundamental matrix F such that $\mathbf{x}_i' F \mathbf{x}_i = 0$

Algorithm

Normalize the image by $\hat{\mathbf{x}}_i = T\mathbf{x}_i$ and $\hat{\mathbf{x}}_i' = T'\mathbf{x}_i'$

Find centroid of points in each image, determine the range, and normalize all points between 0 and 1

Linear solution

determining the eigen vector corresponding to the smallest eigen value of A,

$$Af = \begin{bmatrix} \hat{x}_1'\hat{x}_1 & \hat{x}_1'\hat{y}_1 & \hat{x}_1' & \hat{y}_1'\hat{x}_1 & \hat{y}_1'\hat{y}_1 & \hat{y}_1' & \hat{x}_1 & \hat{y}_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{x}_8'\hat{x}_8 & \hat{x}_8'\hat{y}_8 & \hat{x}_8' & \hat{y}_8'\hat{x}_8 & \hat{y}_8'\hat{y}_8 & \hat{y}_8' & \hat{x}_8' & \hat{y}_8 & 1 \end{bmatrix} f = 0$$

Normalized 8-point algorithm

Construct

$$\hat{F} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$$

Normalize

$$\hat{F} = \hat{F} / \| \hat{F} \|$$

Constraint enforcement SVD decomposition

$$\hat{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V' \qquad (\sigma_1 \geq \sigma_2 \geq \sigma_3)$$

Rank enforcement

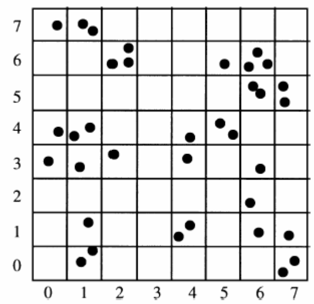
$$\tilde{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V' \qquad (\sigma_3 = 0)$$

De-normalization: $F = T'^T \tilde{F} T$ (3) Show this

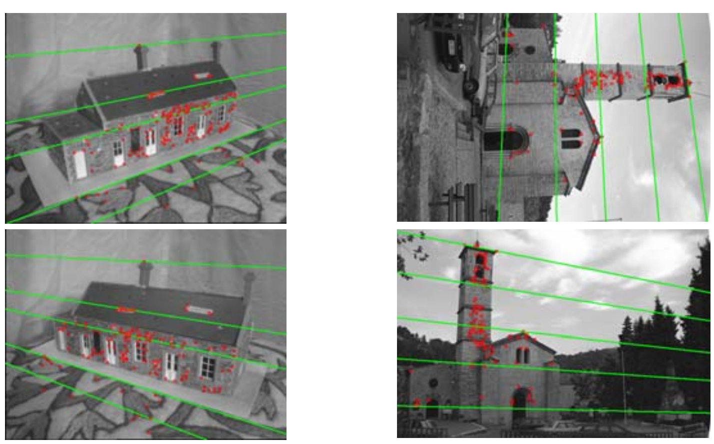
$$T = \begin{bmatrix} a_x & 0 & d_x \\ 0 & a_y & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Robust Fundamental Matrix Estimation (by Zhang)

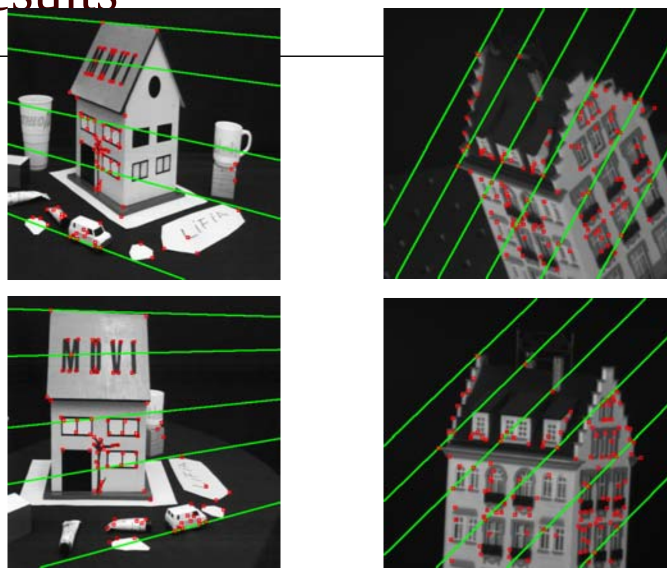
- Uniformly divide the image into 8×8 grid.
- Randomly select 8 grid cells and pick one pair of corresponding points from each grid cell, then use Hartley's 8-point algorithm to compute Fundamental Matrix F_i.
- For each F_i, compute the median of the squared residuals R_i.
  - $R_i = \text{median}_k[d(p_{1k}, F_i p_{2k}) + d(p_{2k}, F'_i p_{1k})]$
- Select the best F_i according to R_i.
- Determine outliers if $R_k > Th$.
- Using the remaining points compute the fundamental Matrix F by weighted least square method.



Results



Results



Homework Due 1/23/2008

- Show (1), (2) and (3).
- Experiment with five different opencv routines.