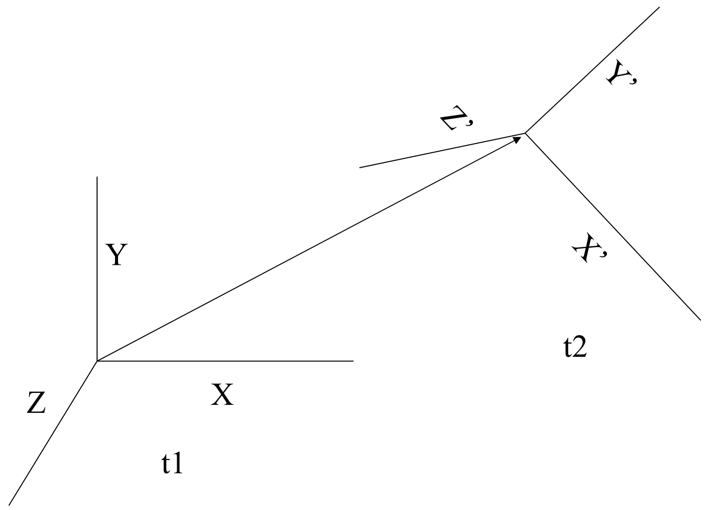
PART I

Measurement of Motion

Contents

- Image Motion Models
- Optical Flow Methods
  - Horn & Schunck
  - Lucas and Kanade
  - Anandan et al
  - Szeliski
  - Mann & Picard
- Video Mosaics

3-D Rigid Motion



3-D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

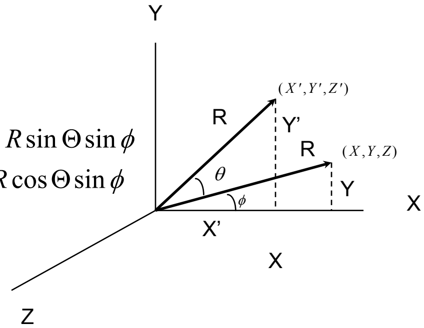
Rotation matrix (9 unknowns)

Translation (3 unknowns)

Rotation

$$X = R\cos\phi$$

$$Y = R\sin\phi$$



$$X' = R\cos(\Theta+\phi) = R\cos\Theta\cos\phi - R\sin\Theta\sin\phi$$
$$Y' = R\sin(\Theta+\phi) = R\sin\Theta\cos\phi + R\cos\Theta\sin\phi$$

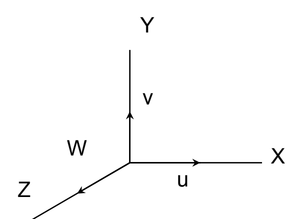
$$X' = X\cos\Theta - Y\sin\Theta$$

$$Y' = X\sin\Theta + Y\cos\Theta$$

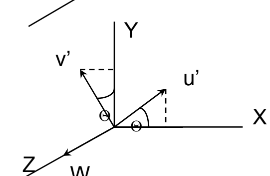
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Rotation (continued)

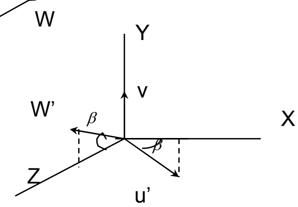
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$



Euler Angles

$$R = R_Z^{\alpha} R_Y^{\beta} R_X^{\gamma} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

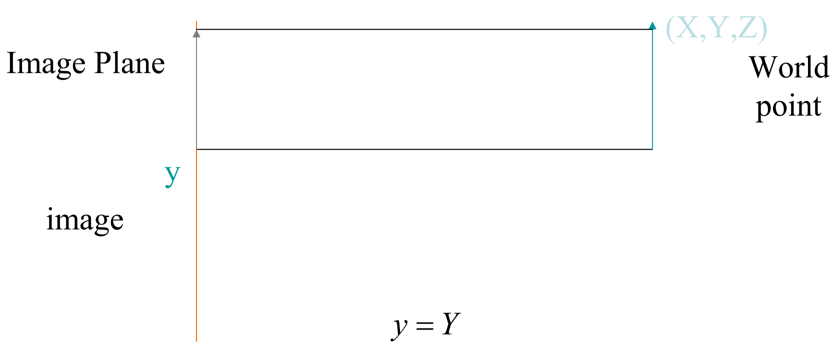
if angles are small($\cos\Theta \approx 1 \quad \sin\Theta \approx \Theta$

$$R = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix}$$

Image Motion Models

Displacement Model

Image Formation: Orthographic Projection



$$y = Y$$
$$x = X$$

Orthographic Projection

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$x = X$$
$$y = Y$$

(x,y)=image coordinates, (X,Y,Z)=world coordinates

$$x' = r_{11}x + r_{12}y + (r_{13}Z + T_X)$$
$$y' = r_{21}x + r_{22}y + (r_{23}Z + T_Y)$$

$$x' = a_1 x + a_2 y + b_1$$
$$y' = a_3 x + a_4 y + b_2$$

Affine Transformation

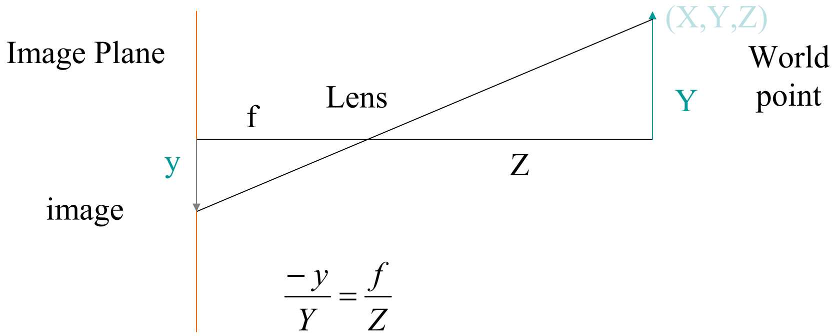
$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Orthographic Projection (contd.)

$$\begin{bmatrix} X' \\ Y \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$x' = x - \alpha y + \beta Z + T_X$$
$$y' = \alpha x + y - \gamma Z + T_Y$$

Image Formation: Perspective Projection



$$\frac{-y}{Y} = \frac{f}{Z}$$

$$y = -\frac{fY}{Z} \qquad x = -\frac{fX}{Z}$$

Perspective Projection

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$X' = r_{11}X + r_{12}Y + r_{13}Z + T_X$$
$$Y' = r_{21}X + r_{22}Y + r_{23}Z + T_Y$$
$$Z' = r_{31}X + r_{32}Y + r_{33}Z + T_Z$$

focal length = -1

$$x' = \frac{X'}{Z'} \qquad y' = \frac{Y'}{Z'}$$

$$x' = \frac{r_{11}x + r_{12}y + r_{13} + \frac{T_X}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}}$$ ← scale ambiguity

$$y' = \frac{r_{21}x + r_{22}y + r_{23} + \frac{T_Y}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}}$$

Plane+Perspective(projective)

$$aX+bY+cZ=1$$

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 1$$

equation of a plane

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T$$

3d rigid motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Plane+Perspective(projective)

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = A\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

focal length = -1

$$X'=a_1X+a_2Y+a_3Z$$
$$Y'=a_4X+a_5Y+a_6Z$$
$$Z'=a_7X+a_8Y+a_9Z$$

$$x' = \frac{a_1X + a_2Y + a_3Z}{a_7X + a_8Y + a_9Z}$$

$$y' = \frac{a_4X + a_5Y + a_6Z}{a_7X + a_8Y + a_9Z}$$

$$x' = \frac{a_1x + a_2y + a_3}{a_7x + a_8y + a_9}$$

$$y' = \frac{a_4x + a_5y + a_6}{a_7x + a_8y + a_9}$$

$$a_9 = 1$$

scale ambiguity

Plane+perspective (contd.)

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}$$

$$\mathbf{X}' = \frac{\mathbf{A}\mathbf{X} + b}{C^T \mathbf{X} + 1}$$

$$\mathbf{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix},$$

$$b = \begin{bmatrix} a_3 \\ a_6 \end{bmatrix}, C = \begin{bmatrix} a_7 \\ a_8 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Projective
Or
Homography

Least Squares

- Eq of a line

$$mx + c = y$$

  - Consider n points

$$mx_1 + c = y_1$$
$$\vdots$$
$$mx_n + c = y_n$$

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ & \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \\ y_n \end{bmatrix}$$

$$\mathbf{A}p = Y$$

Least Squares Fit

$$\mathbf{A}p = Y$$

$$\mathbf{A}^{\mathbf{T}}\mathbf{A}p = \mathbf{A}^{\mathbf{T}}Y$$

$$p = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}Y$$

$$\Uparrow$$

$$\min \sum_{i=1}^{n} (y_i - m x_i - c)^2$$

Determining Projective transformation using point correspondences

- If point correspondences (x,y)<-->(x',y') are known
- a's can be determined by least squares fit

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}$$

$$a_7 x' x + a_8 x' y + x' = a_1 x + a_2 y + a_3$$

$$a_7 y' x + a_8 y' y + y' = a_7 x + a_8 y + a_6$$

$$a_1 x + a_2 y + a_3 - a_7 x' x - a_8 x' y = x'$$

$$a_7 x + a_8 y + a_6 - a_7 y' x - a_8 y' y = y'$$

Two rows for each point i

$$\begin{bmatrix} & & & & \vdots & & & \\ x_i & y_i & 1 & 0 & 0 & 0 & -x_i x_i' & -y_i x_i' \\ 0 & 0 & 0 & x_i & y_i & 1 & -x_i y_i' & -y_i y_i' \\ & & & & \vdots & & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix}$$

Summary of Displacement Models

Translation

$$x' = x + b_1$$
$$y' = y + b_2$$

Biquadratic

$$x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$$
$$y' = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} y^2 a_{12} xy$$

Rigid

$$x' = x\cos\theta - y\sin\theta + b_1$$
$$y' = x\sin\theta + y\cos\theta + b_2$$

Bilinear

$$x' = a_1 + a_2 x + a_3 y + a_4 xy$$
$$y' = a_5 + a_6 x + a_7 y + a_8 xy$$

Affine

$$x' = a_1 x + a_2 y + b_1$$
$$y' = a_3 x + a_4 y + b_2$$

Pseudo Perspective

$$x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$$
$$y' = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2$$

Projective

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$
$$y' = \frac{a_3 x + a_4 y + b_1}{c_1 x + c_2 y + 1}$$

Displacement Models (contd)

- Translation
  - simple
  - used in block matching
  - no zoom, no rotation, no pan and tilt
- Rigid
  - rotation and translation
  - no zoom, no pan and tilt

Displacement Models (contd)

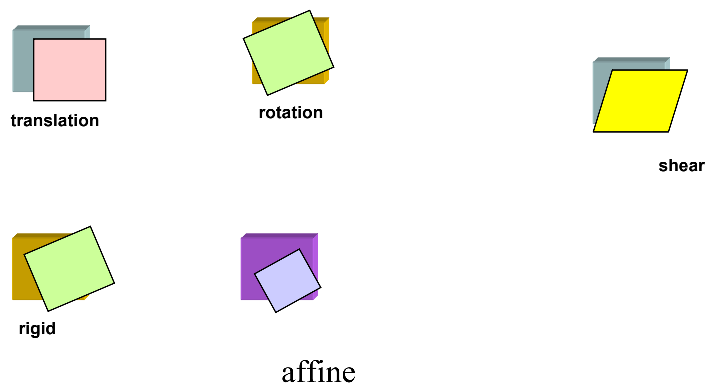
- Affine
  - rotation about optical axis only
  - can not capture pan and tilt
  - orthographic projection

- Projective
  - exact eight parameters (3 rotations, 3 translations and 2 scalings)
  - difficult to estimate (due to denominator terms)

Displacement Models (contd)

- Biquadratic
  - obtained by second order Taylor series
  - 12 parameters
- Bilinear
  - obtained from biquadratic model by removing square terms
  - most widely used
  - not related to any physical 3D motion
- Pseudo-perspective
  - obtained by removing two square terms and constraining four remaining to 2 degrees of freedom

Spatial Transformations



Affine Matrix Decomposition

$$
\begin{aligned}
A &= UDV^T = U(V^T V)DV = (UV^T)(VDV^T) \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_1 & v_h \\ v_h & v_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \begin{bmatrix} 1/\kappa & 0 \\ 0 & 1/\kappa \end{bmatrix} \begin{bmatrix} v_1 & v_h \\ v_h & v_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \begin{bmatrix} v_1/\kappa & v_h \\ v_h & v_2/\kappa \end{bmatrix} \\
&= R(\alpha)S(\kappa)E
\end{aligned}
$$

Decomposition of Affine

$$A = SVD = S(DD^{-1})VD = (SD)(D^{-1}VD)$$
$$= R(\alpha)C = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_1 & v_h \\ v_h & v_2 \end{bmatrix}$$

$$A = \Delta \begin{bmatrix} \frac{1}{\rho} & 0 \\ \beta & \rho \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad A = \begin{bmatrix} \frac{s_y}{\sqrt{s_x s_y}\beta} & 0 \\ & s_x \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Delta = scale\_factor = \sqrt{s_x s_y}, \quad \rho = scale\_ratio = \sqrt{\frac{s_x}{s_y}}, \beta = \text{skew}$$