Lecture-14

Kalman Filter

Main Points

- Very useful tool.
- It produces an optimal estimate of the state vector based on the noisy measurements (observations).
- For the state vector it also provides confidence (certainty) measure in terms of a covariance matrix.
- It integrates an estimate of state over time.
- It is a sequential state estimator.

State-Space Model

State-transition equation

State model error
With covariance
Q(k)

$$\mathbf{z}(k) = \Phi(k, k-1)\mathbf{z}(k-1) + \mathbf{w}(k)$$

State Vector

Measurement (observation) equation

$$\mathbf{y}(k) = \mathbf{H}(k)\mathbf{z}(k) + \mathbf{v}(k)$$

Observation
Noise with covarianc
R(k)

Measurement Vector

Kalman Filter Equations

State Prediction $\hat{\mathbf{z}}_b(k) = \Phi(k, k-1)\hat{\mathbf{z}}_a(k-1)$

Covariance Prediction $\mathbf{P}_b(k) = \Phi(k, k-1)\mathbf{P}_a(k-1)\Phi^T(k, k-1) + \mathbf{Q}(k)$

Kalman Gain $\mathbf{K}(k) = \mathbf{P}_b(k)\mathbf{H}^T(k)(\mathbf{H}(k)\mathbf{P}_b(k)\mathbf{H}^T(k) + \mathbf{R}(k))^{-1}$

State-update $\hat{\mathbf{z}}_a(k) = \hat{\mathbf{z}}_b(k) + \mathbf{K}(k)[\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{z}}_b(k)]$

Covariance-update $\mathbf{P}_a(k) = \mathbf{P}_b(k) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}_b(k)$

Two Special Cases

- Steady State

$$\Phi(k,k-1)=\Phi$$
$$\mathbf{Q}(k)=\mathbf{Q}$$
$$\mathbf{H}(k)=\mathbf{H}$$
$$\mathbf{R}(k)=\mathbf{R}$$

- Recursive least squares

$$\Phi(k,k-1)=\mathbf{I}$$
$$\mathbf{Q}(k)=0$$

Comments

- ☐ In some cases, state transition equation and the observation equation both may be non-linear.
- ☐ We need to linearize these equation using Taylor series.

Extended Kalman Filter

$$\mathbf{z}(k)=\mathbf{f}(\mathbf{z}(k-1))+\mathbf{w}(k)$$

$$\mathbf{y}(k)=\mathbf{h}(\mathbf{z}(k))+\mathbf{v}(k)$$

$$\mathbf{f}(\mathbf{z}(k-1))\approx\mathbf{f}(\hat{\mathbf{z}}_a(k-1))+\frac{\partial\mathbf{f}(\mathbf{z}(k-1))}{\partial\mathbf{z}(k-1)}(\mathbf{z}(\mathrm{k}-1)-\hat{\mathbf{z}}_a(k-1))$$

Taylor series

$$\mathbf{h}(\mathbf{z}(k))\approx\mathbf{h}(\hat{\mathbf{z}}_b(k))+\frac{\partial\mathbf{h}(\mathbf{z}(k))}{\partial\mathbf{z}(k)}(\mathbf{z}(k)-\hat{\mathbf{z}}_b(k-1))$$

Extended Kalman Filter

$$\mathbf{z}(k)=\mathbf{f}(\mathbf{z}(k-1))+\mathbf{w}(k)$$

$$\mathbf{z}(k)=\mathbf{f}(\hat{\mathbf{z}}_a(k-1))+\frac{\partial\mathbf{f}(\mathbf{z}(k-1))}{\partial\mathbf{z}(k-1)}(\mathbf{z}(\mathrm{k}-1)-\hat{\mathbf{z}}_a(k-1))+\mathbf{w}(k)$$

$$\mathbf{z}(k)\approx\Phi(k,k-1)\mathbf{z}(k-1)+\mathbf{u}(k)+\mathbf{w}(k)$$

$$\mathbf{u}(k)=\mathbf{f}(\hat{\mathbf{z}}_a(k-1))-\Phi(k,k-1)\hat{\mathbf{z}}_a(k-1)$$

$$\Phi(k,k-1)=\frac{\partial\mathbf{f}(\mathbf{z}(k-1))}{\partial\mathbf{z}(k-1)}$$

Extended Kalman Filter

$$\mathbf{y}(k) = \mathbf{h}(\mathbf{z}(k)) + \mathbf{v}(k)$$

$$\mathbf{y}(k) = \mathbf{h}(\hat{\mathbf{z}}_b(k)) + \frac{\partial \mathbf{h}(\mathbf{z}(k))}{\partial \mathbf{z}(k)}(\mathbf{z}(k) - \hat{\mathbf{z}}_b(k-1)) + \mathbf{v}(k)$$

$$\tilde{\mathbf{y}}(k) \approx \mathbf{H}(k)\mathbf{z}(k) + \mathbf{v}(k)$$

$$\tilde{\mathbf{y}}(k) = \mathbf{y}(k) - \mathbf{h}(\hat{\mathbf{z}}_b(k)) + \mathbf{H}(k)\hat{\mathbf{z}}_b(k)$$

$$\mathbf{H}(k) = \frac{\partial \mathbf{h}(\mathbf{z}(k))}{\partial \mathbf{z}(k)}$$

Multi-Frame Feature Tracking

Application of Kalman Filter

- Assume feature points have been detected in each frame.
- We want to track features in multiple frames.
- Kalman filter can estimate the position and uncertainty of feature in the next frame.
  - Where to look for a feature
  - how large a region should be searched

$\mathbf{p}_k = [x_k, y_k]^T$ Location

$\mathbf{v}_k = [u_k, v_k]^T$ Velocity

$\mathbf{Z} = [x_k, y_k, u_k, v_k]^T$ State Vector

System Model

$\mathbf{p}_k = [x_k, y_k]^T$ $\quad \mathbf{p}_k = \mathbf{p}_{k-1} + \mathbf{v}_{k-1} + \xi_{k-1}$

$\mathbf{v}_k = [u_k, v_k]^T$ $\quad \mathbf{v}_k = \mathbf{v}_{k-1} + \eta_{k-1}$

noise

$\mathbf{Z} = [x_k, y_k, u_k, v_k]^T$ $\quad \mathbf{Z}_k = \Phi_{k-1}\mathbf{Z}_{k-1} + \mathbf{w}_{k-1}$

$$\Phi_{k-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{w}_{k-1} = \begin{bmatrix} \xi_{k-1} \\ \eta_{k-1} \end{bmatrix}$$

Measurement Model

$\mathbf{p}_k = [x_k, y_k]^T$

$\mathbf{v}_k = [u_k, v_k]^T$

$\mathbf{Z} = [x_k, y_k, u_k, v_k]^T$

$$\mathbf{y}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k \\ \mathbf{v}_k \end{bmatrix} + \mu_k$$

$$\mathbf{y}_k = \mathbf{H} \begin{bmatrix} \mathbf{p}_k \\ \mathbf{v}_k \end{bmatrix} + \mu_k$$

Measurement matrix

Kalman Filter Equations

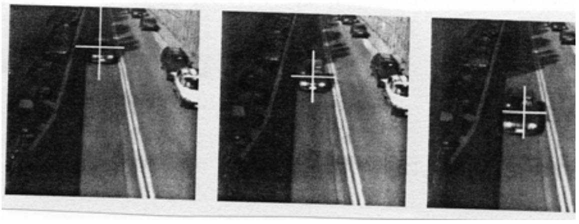
State Prediction $\quad \hat{\mathbf{z}}_b(k) = \Phi(k, k-1)\hat{\mathbf{z}}_a(k-1)$

Covariance Prediction $\quad \mathbf{P}_b(k) = \Phi(k, k-1)\mathbf{P}_a(k-1)\Phi^T(k, k-1) + \mathbf{Q}(k)$

Kalman Gain $\quad \mathbf{K}(k) = \mathbf{P}_b(k)\mathbf{H}^T(k)(\mathbf{H}(k)\mathbf{P}_b(k)\mathbf{H}^T(k) + \mathbf{R}(k))^{-1}$

State-update $\quad \hat{\mathbf{z}}_a(k) = \hat{\mathbf{z}}_b(k) + \mathbf{K}(k)[\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{z}}_b(k)]$

Covariance-update $\quad \mathbf{P}_a(k) = \mathbf{P}_b(k) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}_b(k)$



Kalman Filter: Relation to Least Squares

$$f_i(\mathbf{Z},\mathbf{y}_i)=0$$

$$\Downarrow$$

Taylor series

$$f_i(\mathbf{Z},\mathbf{y}_i)=0\approx f_i(\hat{\mathbf{Z}}_{i-1},\hat{\mathbf{y}}_i)+\frac{\partial f_i}{\partial \mathbf{y}}(\mathbf{y}-\hat{\mathbf{y}}_i)+\frac{\partial f_i}{\partial \mathbf{z}}(\mathbf{z}-\hat{\mathbf{z}}_{\mathbf{i}})+w_i$$

$$\mathbf{Y}_i=H_i\mathbf{Z}+w_i$$

$$\mathbf{Y}_i=-f_i(\hat{\mathbf{Z}}_{i-1},\hat{\mathbf{y}}_i)+\frac{\partial f_i}{\partial \mathbf{z}}\hat{\mathbf{z}}_{i-1},\ H_i=\frac{\partial f_i}{\partial \mathbf{z}}$$

$$w_i=\frac{\partial f_i}{\partial \mathbf{y}}(\mathbf{y}-\hat{\mathbf{y}}_i)$$

Kalman Filter: Relation to Least Squares

Estimate state such that the following is minimized:
-first term: initial estimate weighted by corresponding covariance
-second term: other measurements weighted by corresponding covariances

$$C=(\hat{\mathbf{Z}}_0-\mathbf{Z})^T P_0^{-1}(\hat{\mathbf{Z}}_0-\mathbf{Z})+\sum_{i=1}^{k}(\mathbf{Y}_i-H_i\mathbf{Z})^T W^{-1}{}_i(\mathbf{Y}_i-H_i\mathbf{Z})$$

minimize

$$\Downarrow$$

$$\hat{\mathbf{Z}}=[P_0^{-1}+\sum_{i=1}^{k}H_i{}^TW_i^{-1}H_i]^{-1}[P_0^{-1}\hat{\mathbf{Z}}_0+\sum_{i=1}^{k}H_i{}^TW_i^{-1}\mathbf{Y}_i]$$

Batch Mode

Kalman Filter: Relation to Least Squares

$$\hat{\mathbf{Z}}_k=[P_0^{-1}+\sum_{i=1}^{k}H_i{}^TW_i^{-1}H_i]^{-1}[P_0^{-1}\hat{\mathbf{Z}}_0+\sum_{i=1}^{k}H_i{}^TW_i^{-1}\mathbf{Y}_i]$$

$$\hat{\mathbf{Z}}_{k-1}=[P_0^{-1}+\sum_{i=1}^{k-1}H_i{}^TW_i^{-1}H_i]^{-1}[P_0^{-1}\hat{\mathbf{Z}}_0+\sum_{i=1}^{k-1}H_i{}^TW_i^{-1}\mathbf{Y}_i]$$

Recursive Mode

Kalman Filter: Relation to Least Squares

$$\mathbf{Z}_k=\mathbf{Z}_{k-1}+K_k(Y_k-H_k\mathbf{Z}_{k-1})$$

$$K_k=P_{k-1}H^T{}_k\ (W_k+H_kP_{k-1}H_k{}^T)^{-1}$$

$$P_k=(I-K_kH_k)P_{k-1}$$

$$\Phi(k,k-1)=\mathbf{I}$$

$$\mathbf{Q}(k)=0$$

$$Y_k=-f^T(\mathbf{Z}_{k-1},\mathbf{y}_{k-1})+\frac{\partial f}{\partial \mathbf{Z}}\mathbf{Z}_{k-1}$$

$$H_k=\frac{\partial f}{\partial \mathbf{Z}}$$

$$W^k=\frac{\partial f}{\partial \mathbf{y}}\mathrm{A}_k\frac{\partial f}{\partial \mathbf{y}}^T$$

Covariance matrix for measurement
Vector *y*

Kalman Filter (Least Squares)

State Prediction $\hat{\mathbf{z}}_b(k) = \Phi(k, k-1)\hat{\mathbf{z}}_a(k-1)$

$$\hat{\mathbf{z}}_b(k) = \hat{\mathbf{z}}_a(k-1)$$

Covariance Prediction $\mathbf{P}_b(k) = \Phi(k, k-1)\mathbf{P}_a(k-1)\Phi^T(k, k-1) + \mathbf{Q}(k)$

$$\mathbf{P}_b(k) = \mathbf{P}_a(k-1)$$

Kalman Gain

$$\mathbf{K}(k) = \mathbf{P}_b(k)\mathbf{H}^T(k)(\mathbf{H}(k)\mathbf{P}_b(k)\mathbf{H}^T(k) + \mathbf{R}(k))^{-1}$$

$$\mathbf{K}(k) = \mathbf{P}_b(k)\mathbf{H}^T(k)(\mathbf{H}(k)\mathbf{P}_b(k)\mathbf{H}^T(k) + \mathbf{W}(k))^{-1}$$

Kalman Filter (Least Squares)

State-update $\hat{\mathbf{z}}_a(k) = \hat{\mathbf{z}}_b(k) + \mathbf{K}(k)[\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{z}}_b(k)]$

$$\hat{\mathbf{z}}\ (k) = \hat{\mathbf{z}}\ (k-1) + \mathbf{K}(k)[\mathbf{y}(k) - \mathbf{H}(k)\hat{\mathbf{z}}\ (k-1)]$$

Covariance-update

$$\mathbf{P}_a(k) = \mathbf{P}_b(k) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}_b(k)$$

$$\mathbf{P}\ (k) = \mathbf{P}\ (k-1) - \mathbf{K}(k)\mathbf{H}(k)\mathbf{P}\ (k-1)$$

Computing Motion Trajectories

Algorithm For Computing Motion Trajectories

- ☐ Compute tokens using Moravec’s interest operator (intensity constraint).
- ☐ Remove tokens which are not interesting with respect to motion (optical flow constraint).
  - Optical flow of a token should differ from the mean optical flow around a small neighborhood.

Algorithm For Computing Motion Trajectories

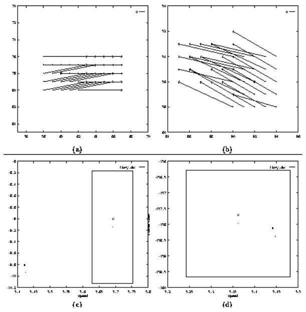
- Link optical flows of a token in different frames to obtain motion trajectories.
  - Use optical flow at a token to predict its location in the next frame.
  - Search in a small neighborhood around the predicted location in the next frame for a token.
- Smooth motion trajectories using Kalman filter.

Kalman Filter (Ballistic Model)

$$x(t) = .5a_x t^2 + v_x t + x_0 \qquad \mathbf{Z} = (a_x, a_y, v_x, v_y)$$

$$y(t) = .5a_y t^2 + v_y t + y_0 \qquad \mathbf{y} = (x(t), y(t))$$

$$f(\mathbf{Z}, \mathbf{y}) = (x(t) - .5a_x t^2 - v_x t - x_0, y(t) - .5a_y t^2 - v_y t - y_0)$$



Kalman Filter (Ballistic Model)

$$\mathbf{Z}(k) = \mathbf{Z}(k-1) + K(k)(Y(k) - H(k)\mathbf{Z}(k-1))$$

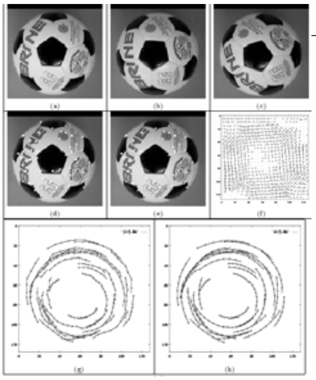
$$K(k) = P(k-1)H^T(k) \; (W(k) + H^T P(k-1)H^T(k))^{-1}$$

$$P(k) = (I - K(k)H(k))P(k-1)$$

$$Y(k) = -f^T(\mathbf{Z}(\mathbf{k-1}), \mathbf{y}) + \frac{\partial f}{\partial \mathbf{Z}}\mathbf{Z}(k-1)$$

$$H(k) = \frac{\partial f}{\partial \mathbf{Z}}$$

$$W(k) = \frac{\partial f}{\partial \mathbf{y}} \mathrm{A}(k) \frac{\partial f}{\partial \mathbf{y}}^T$$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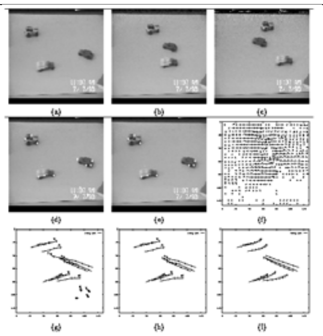
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