$\mathbf{X}_0 = (X_0, Y_0, Z_0)^\top$: 3-D coordinates of a point at time t_0

$\mathbf{T} = (T_1, T_2, T_3)^\top$: Translation vector

spherical coordinates: slant, θ_T, and tilt, ϕ_T:

$$\mathbf{T} = (\sin\theta_T\cos\phi_T, \sin\theta_T\sin\phi_T, \cos\theta_T)^\top. \tag{1}$$

$\mathbf{q} = (q_0, q_1, q_2, q_3)^\top$: Quaternion

$\mathbf{q} = (\sin\frac{\alpha}{2}n, \cos\frac{\alpha}{2})^\top$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 + q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{2}$$

$$\mathbf{q}(t) = \left(\frac{w_x}{w}\sin\frac{wt}{2}, \quad \frac{w_y}{w}\sin\frac{wt}{2}, \quad \frac{w_z}{w}\sin\frac{wt}{2}, \quad \cos\frac{wt}{2} \right)^\top, \tag{3}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

3-D motion:

$$\mathbf{X_t} = \mathbf{R}(\mathbf{q}(t))\mathbf{X_0} + t\mathbf{T}. \tag{4}$$

Perspective projection:

$\mathbf{X} = (X_1, X_2, X_3),$

$x = f\frac{X_1}{X_3}, y = f\frac{X_2}{X_3}$

Unknowns: $\mathbf{a} = (\theta_T, \phi_T, w_x, w_y, w_z, Z_0)^\top.$

Problem: determine $\mathbf{a}$, given $\{\mathbf{x_s}, \mathbf{x_{s+1}}, \cdots, \mathbf{x_{s+N}}\}$,

where $\mathbf{x_t} = (x_t, y_t)^\top, t = s, \cdots, s + N, s \geq 0$

Let $\mathbf{h_t}(\mathbf{a})$ be the 3-D coordinates of a point at time t, **given parameter a**:

$$\mathbf{h_t}(\mathbf{a}) = \mathbf{R}(\mathbf{q}(t))\mathbf{X_0} + t\mathbf{T}. \tag{5}$$

Given image coordinates $\mathbf{x_t}$, and depth Z_t, its corresponding 3-D coordinates are:

$\mathbf{X_t} = (Z_t x_t/f, Z_t y_t/f, Z_t)^\top.$

Let $\mathbf{p} = (0, 0, 1)$, then $Z_t = \mathbf{p}\,\mathbf{h_t}(\mathbf{a})$. Now, $\mathbf{X_t}$ becomes

$$\mathbf{X_t} = \mathbf{p}\,\mathbf{h_t}(\mathbf{a}) \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix}.$$

The objective function to be minimized is given by:

$$E^2(\mathbf{a}) = \sum_{t=s}^{s+N} \| \mathbf{X_t} - \mathbf{h_t}(\mathbf{a}) \|^2, \tag{6}$$

$$\mathbf{X_t} = \mathbf{p}\ \mathbf{h_t}(\mathbf{a}) \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix}.$$

$$\mathbf{h_t}(\mathbf{a}) = \mathbf{R}(\mathbf{q}(t))\mathbf{X_0} + t\mathbf{T}. \tag{7}$$

$$(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})) = ((\mathbf{p}\ \mathbf{h_t}(\mathbf{a})) \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix} - \mathbf{h_t}(\mathbf{a})) \tag{8}$$

$$\frac{\partial}{\partial \mathbf{a_i}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})) = t\mathbf{p}\ \frac{\partial \mathbf{T}}{\partial \mathbf{a_i}} \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix} - t\frac{\partial \mathbf{T}}{\partial \mathbf{a_i}}, \quad i = 1, 2, \tag{9}$$

$$\frac{\partial}{\partial \mathbf{a_i}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})) = \mathbf{p}\ \frac{\partial \mathbf{R}}{\partial \mathbf{a_i}}\ \mathbf{X_0} \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix} - \frac{\partial \mathbf{R}}{\partial \mathbf{a_i}}\ \mathbf{X_0}, \quad i = 3, 4, 5, \tag{10}$$

$$\frac{\partial}{\partial \mathbf{a_6}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})) = \left\{ \mathbf{pR} \begin{pmatrix} x_0/f \\ y_0/f \\ 1 \end{pmatrix} \right\} \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix} - \mathbf{R} \begin{pmatrix} x_0/f \\ y_0/f \\ 1 \end{pmatrix}, \tag{11}$$

$$\mathbf{T} = (\sin\theta_T\cos\phi_T, \sin\theta_T\sin\phi_T, \cos\theta_T)^\top. \tag{12}$$

where

$$\frac{\partial \mathbf{T}}{\partial \mathbf{a_1}} = \frac{\partial \mathbf{T}}{\partial \theta_T} = \begin{pmatrix} \cos\theta_T \ \cos\phi_T \\[2mm] \cos\theta_T \ \sin\phi_T \\[2mm] -\sin\theta_T \end{pmatrix}, \quad \frac{\partial \mathbf{T}}{\partial \mathbf{a_2}} = \frac{\partial \mathbf{T}}{\partial \phi_T} = \begin{pmatrix} -\sin\theta_T \ \sin\phi_T \\[2mm] \sin\theta_T \ \cos\phi_T \\[2mm] 0 \end{pmatrix},$$

$\mathbf{q} = (q_0, q_1, q_2, q_3)^\top$: Quaternion

$$\mathbf{q}(t) = \left(\quad \frac{w_x}{w} \sin \frac{wt}{2}, \quad \frac{w_y}{w} \sin \frac{wt}{2}, \quad \frac{w_z}{w} \sin \frac{wt}{2}, \quad \cos \frac{wt}{2} \quad \right)^\top, \tag{13}$$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0 q_1 + q_2 q_3) & 2(q_0 q_2 - q_1 q_3) \\\\ 2(q_0 q_1 - q_2 q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1 q_2 + q_0 q_3) \\\\ 2(q_0 q_2 + q_1 q_3) & 2(q_1 q_2 - q_0 q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{14}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

$$\frac{\partial \mathbf{R}}{\partial \mathbf{a_3}} = \frac{\partial \mathbf{R}}{\partial w_x} = \sum_{k=0}^{3} \frac{\partial \mathbf{R}}{\partial q_k} \frac{\partial q_k}{\partial w_x} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_x}, \quad \frac{\partial \mathbf{R}}{\partial \mathbf{a_4}} = \frac{\partial \mathbf{R}}{\partial w_y} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_y}, \quad \frac{\partial \mathbf{R}}{\partial \mathbf{a_5}} = \frac{\partial \mathbf{R}}{\partial w_z} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_z},$$

where $D_k = \partial \mathbf{R} / \partial q_k$, $k = 0, 1, 2, 3$, are given by

$$\mathbf{D}_0 = 2 \begin{pmatrix} q_0 & q_1 & q_2 \\\\ q_1 & -q_0 & q_3 \\\\ q_2 & -q_3 & -q_0 \end{pmatrix}, \qquad \mathbf{D}_1 = 2 \begin{pmatrix} -q_1 & q_0 & -q_3 \\\\ q_0 & q_1 & q_2 \\\\ q_3 & q_2 & -q_1 \end{pmatrix},$$

$$\mathbf{D}_2 = 2 \begin{pmatrix} -q_2 & q_3 & q_0 \\\\ -q_3 & -q_2 & q_1 \\\\ q_0 & q_1 & q_2 \end{pmatrix}, \qquad \mathbf{D}_3 = 2 \begin{pmatrix} q_3 & q_2 & -q_1 \\\\ -q_2 & q_3 & q_0 \\\\ q_1 & -q_0 & -q_3 \end{pmatrix}.$$

$\mathbf{q} = (q_0, q_1, q_2, q_3)^\top$: Quaternion

$$\mathbf{q}(t) = \left(\begin{array}{cccc} \frac{w_x}{w} \sin \frac{wt}{2}, & \frac{w_y}{w} \sin \frac{wt}{2}, & \frac{w_z}{w} \sin \frac{wt}{2}, & \cos \frac{wt}{2} \end{array} \right)^\top, \tag{15}$$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0 q_1 + q_2 q_3) & 2(q_0 q_2 - q_1 q_3) \\ \\ 2(q_0 q_1 - q_2 q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1 q_2 + q_0 q_3) \\ \\ 2(q_0 q_2 + q_1 q_3) & 2(q_1 q_2 - q_0 q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{16}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

Now, $\partial q_k / \partial \mathbf{a}_i$, $k = 0, 1, 2, 3$, $i = 3, 4, 5$, are given by

$\frac{\partial q_0}{\partial w_x} = \left(\frac{1}{w} - \frac{w_x^2}{w^3} \right) \sin \frac{wt}{2} + \frac{t}{2} \frac{w_x^2}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_1}{\partial w_x} = -\frac{w_x w_y}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_x w_y}{w^2} \cos \frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_x} = -\frac{w_x w_z}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_x w_z}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_3}{\partial w_x} = -\frac{t}{2} \frac{w_x}{w} \sin \frac{wt}{2}$,

$\frac{\partial q_0}{\partial w_y} = -\frac{w_x w_y}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_x w_y}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_1}{\partial w_y} = \left(\frac{1}{w} - \frac{w_y^2}{w^3} \right) \sin \frac{wt}{2} + \frac{t}{2} \frac{w_y^2}{w^2} \cos \frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_y} = -\frac{w_y w_z}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_y w_z}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_3}{\partial w_y} = -\frac{t}{2} \frac{w_y}{w} \sin \frac{wt}{2}$,

$\frac{\partial q_0}{\partial w_z} = -\frac{w_x w_z}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_x w_z}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_1}{\partial w_z} = -\frac{w_y w_z}{w^3} \sin \frac{wt}{2} + \frac{t}{2} \frac{w_y w_z}{w^2} \cos \frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_z} = \left(\frac{1}{w} - \frac{w_z^2}{w^3} \right) \sin \frac{wt}{2} + \frac{t}{2} \frac{w_z^2}{w^2} \cos \frac{wt}{2}$, $\frac{\partial q_3}{\partial w_z} = -\frac{t}{2} \frac{w_z}{w} \sin \frac{wt}{2}$.

$$E^2(\mathbf{a}) = \sum_{t=s}^{s+N} \| \mathbf{X_t} - \mathbf{h_t}(\mathbf{a}) \|^2, \tag{17}$$

Then the first partial derivatives of the function E^2 with respect to $\mathbf{a_i}$ are given by:

$$\frac{\partial E^2}{\partial \mathbf{a_i}} = 2 \sum_{t=s}^{s+N} (\mathbf{X_t} - \mathbf{h_t}(\mathbf{a}))^\top \frac{\partial}{\partial \mathbf{a_i}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})), \quad i = 1, 2, \cdots, 6. \tag{18}$$

The second partial derivatives are obtained by ignoring the second derivatives of the model

function:

$$\frac{\partial^2 E^2}{\partial \mathbf{a_i} \partial \mathbf{a_j}} \approx 2 \sum_{t=s}^{s+N} \frac{\partial}{\partial \mathbf{a_i}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a}))^\top \frac{\partial}{\partial \mathbf{a_j}}(\mathbf{X_t} - \mathbf{h_t}(\mathbf{a})), \quad i, j = 1, 2, \cdots, 6. \tag{19}$$

Let $\beta_i \stackrel{\text{def}}{=} -\frac{1}{2}\frac{\partial E^2}{\partial a_i}$, and $\alpha_{ij} \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial E^2}{\partial a_i \partial a_j}$, the elements in matrix $[\alpha]$. The minimization problem

is reduced to iteratively solving the following linear equation:

$$\sum_{l=1}^{m} \alpha_{kl} \delta a_l = \beta_k, \tag{20}$$

where m is the number of unknown parameters. Here $m = 6$.

Algorithm: TrajFit

1. Compute $E^2(\mathbf{a})$ in Equation (17). Set $\lambda = 0.001$.

2. Compute β_i and α_{ij}, where $i, j = 1, 2, \cdots, 6$, using equations (18) and (19), respectively.

3. Compute matrix $[\alpha]$ by augmenting its diagonal elements: $\alpha'_{jj} = \alpha_{jj}(1 + \lambda)$, and

   $\alpha'_{jk} = \alpha_{jk}$.

4. Solve Equation (20) for $\delta(\mathbf{a})$ and evaluate $E^2(\mathbf{a} + \delta\mathbf{a})$.

5. If $E^2(\mathbf{a} + \delta\mathbf{a}) \geq \mathrm{E}^2(\mathbf{a})$, increase λ by a factor and go back to 2.

6. If $E^2(\mathbf{a} + \delta\mathbf{a}) < E^2(\mathbf{a})$, decrease λ by a factor, update the trial solution $\mathbf{a} \leftarrow \mathbf{a} + \delta\mathbf{a}$,

   and go back to 2.