Lecture-5

Quadratic Functions

Quadratic Functions

$$f(x) = \frac{1}{2} x^T Q x - b^T x$$ Q is symmetric, Hessian of f

$$\nabla f(x) = Qx - b$$

if x^* is a unique solution of $Qx = b$, then it is a stationary point of f

If the linear system $Qx=b$ can not be solved, then function does not have a stationary point, it is unbounded

Quadratic Functions

$$f(x) = \frac{1}{2}x^T Qx - b^T x \qquad Q \text{ is symmetric, Hessian of } f$$

$$\nabla f(x) = Qx - b$$

According to definition, for any vector *x* and *p:*

$$f(x+\alpha p) = \frac{1}{2}(x+\alpha p)^T Q(x+\alpha p) - b^T(x+\alpha p)$$

Quadratic Functions

$$f(x+\alpha p) = \frac{1}{2}(x+\alpha p)^T Q(x+\alpha p) - b^T(x+\alpha p)$$

$$f(x+\alpha p) = \frac{1}{2}(x^T Q + \alpha p^T Q)(x+\alpha p) - b^T x - b^T \alpha p$$

$$= \frac{1}{2}(x^T Qx + \alpha p^T Qx + x^T Q\alpha p + \alpha^2 p^T Qp) - b^T x - b^T \alpha p$$

$$= \frac{1}{2}x^T Qx - b^T x + \frac{1}{2}(\alpha p^T Qx + x^T Q\alpha p + \alpha^2 p^T Qp) - b^T \alpha p$$

$$f(x+\alpha p) \ = f(x) + \alpha p^T(Qx - b) + \frac{1}{2}\alpha^2 p^T Qp$$

If x^* is stationary point

$$f(x^*+\alpha p) \ = f(x^*) + \alpha p^T(Qx^* - b) + \frac{1}{2}\alpha^2 p^T Qp$$

$$f(x^*+\alpha p) \ = f(x^*) + \frac{1}{2}\alpha^2 p^T Qp$$

Quadratic Functions

$$f(x^* + \alpha p) = f(x^*) + \frac{1}{2}\alpha^2 p^T Q p$$

The behavior of f is determine by matrix Q

Let $Qu_j = \lambda_j u_j$ Eigenvector and eigenvalue

Let p is equal to u_j

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 u_j^T Q u_j$$

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 u_j^T \lambda_j u_j$$

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

Q is symmetric

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

- The change in f when moving away from x^* along the direction u_j depends on the sign of
  - If is positive f will strictly increase as increases
  - If is negative f is decreasing as increases
    - If is zero, the value of f remains constant when moving along any direction parallel to u_j
    - f reduces to a linear function along any such direction, since quadratic term vanishes.

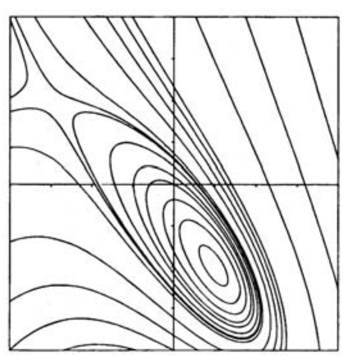
Quadratic Functions

- When all eigenvalues of Q are positive, x^* is the unique global minimum
  - –The contours of f are ellipsoid whose principal axes are in the directions of the eigenvectors of Q, with lengths proportional to square root of corresponding eigenvalues.
- If Q is positive semi-definite, a stationary point (if it exists) is a week local minimum.
- If Q is indefinite and non-singular, x^* is a saddle point, f is unbounded.

$$f(x^* + \alpha u_j) = f(x^*) + \frac{1}{2}\alpha^2 \lambda_j$$

Iso Contours (Contour Map)

$f(x_1, x_2) = c$



$$f(x_1, x_2) = e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1)$$

$c = .2, .4, 1, 1.7, 1.8, 2, 3, 4, 5, 6, 20$

Quadratic Functions

Two positive eigenvalues

$$Q = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} -5.5 \\ -3.5 \end{bmatrix}$$

PD

Eigenvalues 6.8541, 0.1459

Eigenvectors
-0.8507 0.5257
-0.5257 -0.8507

$$f(x) = \frac{1}{2} x^T Q x - b^T x$$

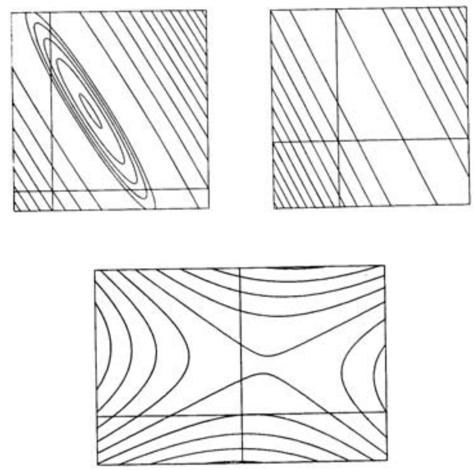


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

One positive eigenvalue,
one zero eigenvalue

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Semi PD

Eigenvalue 5, 0

Eigenvectors
-0.8944 0.4472
-0.4472 -0.8944

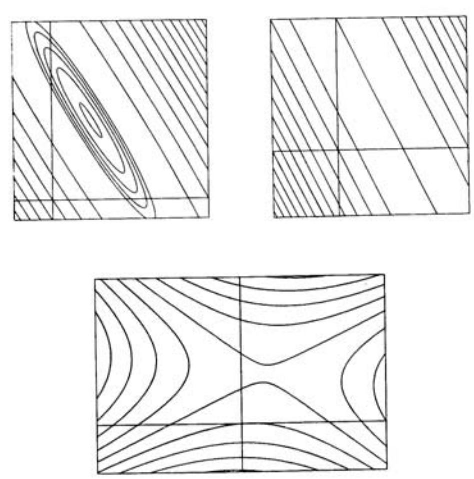


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

One positive eigenvalue,
one negative eigenvalue

$$Q = \begin{bmatrix} 3 & -1 \\ -1 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} -.5 \\ 8.5 \end{bmatrix}$$

Indefinite

Eigenvalue 3.0902, -8.0902

Eigenvectors
-0.9960 -0.0898
0.0898 -0.9960

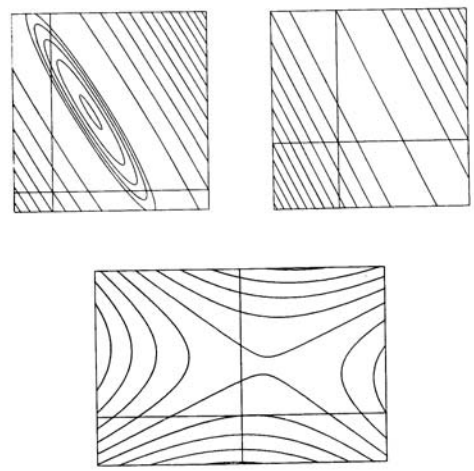


Figure 3f. Contours of: (i) a positive-definite quadratic function; (ii) a positive semi-definite quadratic function; and (iii) an indefinite quadratic function.

Quadratic Functions

How about a function with Q, which is a diagonal matrix?

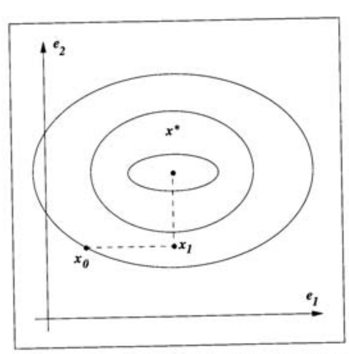
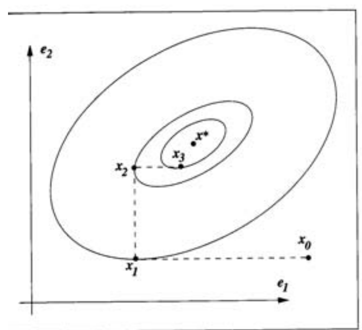


Figure 5.1 Successive minimizations along the coordinate direction
minimizer of a quadratic with a diagonal Hessian in n iterations.

Quadratic Functions



igure 5.2 Successive minimization along coordinate axes does not find the solution ı *n* iterations, for a general convex quadratic.

Quadratic Functions

How about a function with *Q*, which is a multiple of an identity matrix?

Steepest Descent

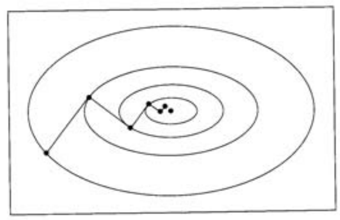


Figure 3.7 Steepest descent steps.

Convergence Rate of Steepest Descent

$$f(x)=\frac{1}{2}x^T Qx-b^T x$$

$$\nabla f(x)=Qx-b$$

x^* is a unique solution of $Qx=b$

Let us compute step length, which minimizes the function:

$$f(x_k-\alpha g_k)=\frac{1}{2}(x_k-\alpha g_k)^T Q(x_k-\alpha g_k)-b^T(x_k-\alpha g_k)$$

Convergence Rate of Steepest Descent

$$\frac{d}{d\alpha} f(x_k - \alpha g_k) = \frac{d}{d\alpha}\left(\frac{1}{2}(x_k - \alpha g_k)^T Q(x_k - \alpha g_k) - b^T(x_k - \alpha g_k)\right) = 0$$

$$= -(x_k - \alpha g_k)^T Q g_k + b^T g_k = 0$$

$$-x_k{}^T Q g_k + \alpha g_k{}^T Q g_k + b^T g_k = 0$$

$$\alpha g_k{}^T Q g_k = x_k{}^T Q g_k - b^T g$$

$$\alpha = \frac{x_k{}^T Q g_k - b^T g_k}{g_k{}^T Q g_k}$$

$$\alpha = \frac{(x_k{}^T Q - b^T) g_k}{g_k Q g_k} \qquad \nabla f(x) = Qx - b$$

$$\alpha = \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k}$$

$$x_{k+1} = x_k - \alpha_k \nabla f_k \qquad x_{k+1} = x_k - \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k} \nabla f_k$$

Convergence Rate of Steepest Descent

Define

$$\frac{1}{2} \| x - x^* \|_Q^2 = f(x) - f(x^*)$$

Using: $x_{k+1} = x_k - \frac{\nabla f_k^T \nabla f_k}{\nabla f_k^T Q \nabla f_k} \nabla f_k$

It can be shown (homework):

$$\| x_{k+1} - x^* \|_Q^2 = \left\{ 1 - \frac{(\nabla f_k^T \nabla f_k)^2}{(\nabla f_k^T Q \nabla f_k)(\nabla f_k^T Q^{-1} \nabla f_k)} \right\} \| x_k - x^* \|_Q^2$$

OR

$$\| x_{k+1} - x^* \|_Q^2 \le \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} \right)^2 \| x_k - x^* \|_Q^2$$

where $0 \le \lambda_1 \le \lambda_2 \le \ldots \lambda_n$ are eigenvalues of Q

Convergence Rate of Steepest Descent

$$\| x_{k+1} - x^* \|_Q^2 \le \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} \right)^2 \| x_k - x^* \|_Q^2$$

As the condition number increases the contours of the quadratic become more elongated, the zigzags of line search becomes more pronounced.

Theorem 3.4: Steepest Descent

$$f(x_{k+1}) - f(x^*) \le \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} \right)^2 (f(x_k) - f(x^*))$$

where $0 \le \lambda_1 \le \lambda_2 \le \ldots \lambda_n$ are eigenvalues of Hessian

If the condition number is 800, and $f(x_1)$=1 and $f(x^*)=0$,
After 1000 iterations the value of function will be .08.