Lecture-4

Line Search Methods: Search Directions, and step lengths

Line Search Methods

$$x_{k+1} \leftarrow x_k + \alpha_k p_k$$

$$p_k \leftarrow -B_k^{-1} \nabla f_k$$

Steepest descent: is an identity matrix

Newton: is a Hessian matrix

Quasi-Newton: is approximation to the Hessian matrix

Inverse Hessian

Instead of inverting approximation of Hessian, we can directly compute the approximation of inverse of Hessian:

$$H_{k+1} = (I - \rho_k s_k y_k^T) H_k (I - \rho_k s_k y_k^T) + \rho_k s_k s_k^T,$$

$$\rho_k = \frac{1}{y_k^T s_k} \qquad s_k = x_{k+1} - x_k, \qquad H_k = B_k^{-1}$$

$$\mathrm{y}_k = \nabla f_{k+1} - \nabla f_k$$

$$p_k = -H_k \nabla f_k$$

Quasi Newton

Conjugate Gradient

$$p_k = -\nabla f(x_k) + \beta_k p_{k-1}$$

is scalar such that
and are conjugate

Two vectors are conjugate with respect to a PD matrix G if

$$p_k{}^T G p_{k-1} = 0$$

Non-interfering directions, with the special property that minimization along one direction is not spoiled by subsequent minimization along another.

Step Length

(Exact Search) The global minimizer of the univariate function:

$$\phi(\alpha) = f(x_k + \alpha p_k) \quad \alpha > 0$$

Too many evaluations of a function, and its gradient

(In-exact search): adequate reduction in f at minimal cost.
Two step method:

- Bracketing (find the interval containing desirable step lengths)
- bisection (compute step length within this interval)

Step Length

Ideal step length is the global minimizer
Step length should achieve sufficient decrease
And it should not be too small

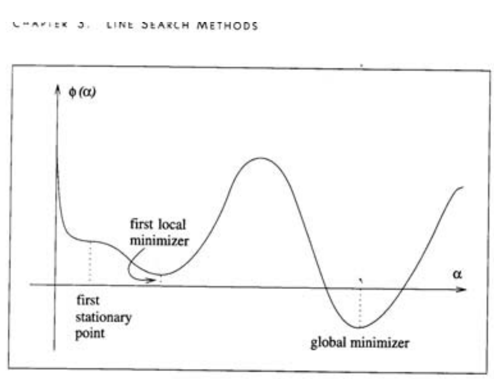


Figure 3.1 The ideal step length is the global minimizer

Simple Condition

Simple condition: reduction in f

$$f(x_k + \alpha p_k) < f(x_k)$$

This is not appropriate.

$$\left\{\frac{5}{k}\right\}, \; k = 1,2,3,\ldots$$

We don not have sufficient reduction

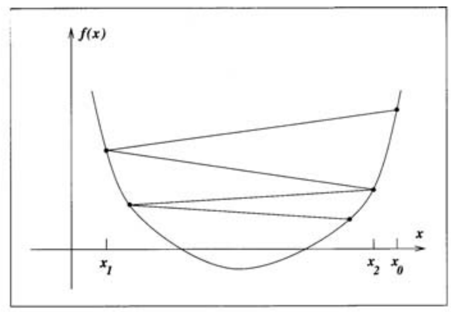


Figure 3.2 Insufficient reduction in f.

Sufficient condition

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \nabla f_k^T p_k, \quad c_1 \in (0,1) \qquad c_1 = 10^{-4}$$

$$-c_1 \alpha \nabla f_k^T p_k \le f(x_k) - f(x_k + \alpha p_k), \quad c_1 \in (0,1)$$

The reduction should be proportional to both the step length, and directional derivative.

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \nabla f_k^T p_k, \quad c_1 \in (0,1)$$
$$f(x_k + \alpha p_k) \le l(\alpha)$$

St line

Sufficient condition

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \nabla f_k^T p_k, \quad c_1 \in (0,1)$$

$$f(x_k + \alpha p_k) \le l(\alpha)$$

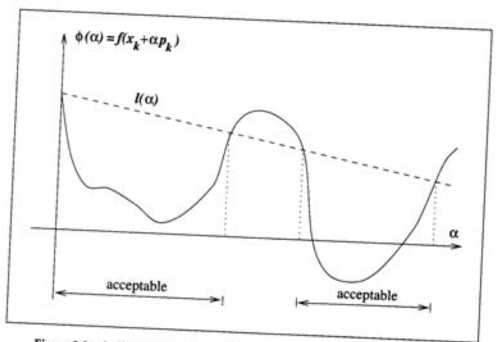


Figure 3.3 Sufficient decrease condition.

Problem:
The sufficient decrease condition is satisfied for all small values of step length

Curvature condition

$$\nabla f(x_k + \alpha p_k)^T p_k \ge c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1, 1)$$

Derivative

$c_2 = .9$ for Newton and Quasi - Newton

$c_2 = .1$ for conjugate gradient

The slope of   is greater than   times the gradient   .

$$\nabla f(x_k + \alpha p_k)^T p_k \geq c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1, 1)$$

Curvature condition

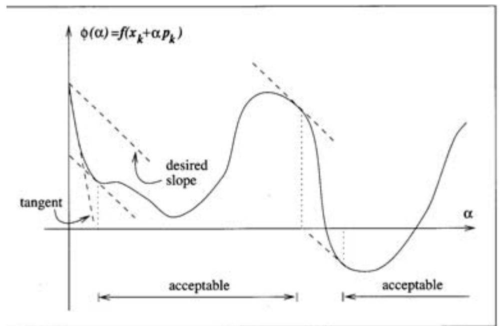
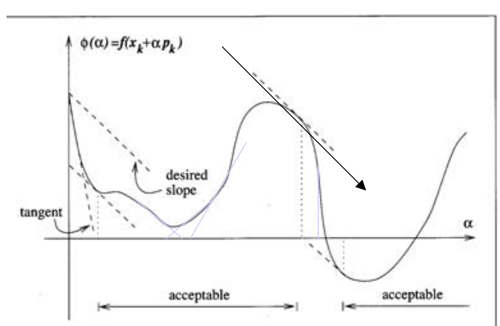


Figure 3.4 The curvature condition.

$$\nabla f(x_k + \alpha p_k)^T p_k \geq c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1, 1)$$

Curvature condition



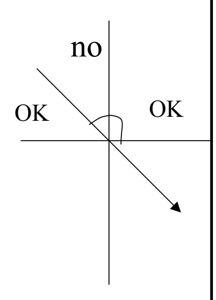


Figure 3.4 The curvature condition.

If the slope is strongly negative (too steep), that means we can re
further along the chosen direction (you should not stop there)
If the slope is positive, it indicates we can not decrease f further in this direction.

Wolfe conditions

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \nabla f_k^T p_k, \quad c_1 \in (0,1)$$ Sufficient decrease

$$\nabla f(x_k + \alpha p_k)^T p_k \ge c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1,1)$$ Curvature

Strong Wolfe conditions

$$f(x_k + \alpha p_k) \le f(x_k) + c_1 \alpha \nabla f_k^T p_k, \quad c_1 \in (0,1)$$

$$|\nabla f(x_k + \alpha p_k)^T p_k| \le c_2 |\nabla f_k^T(x_k) p_k|$$

$$\nabla f(x_k + \alpha p_k)^T p_k \ge c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1,1)$$

This forces step length to lie in at least in a broad neighborhood of a local minimizer or a stationary point of .

should not be too positive, exclude points which are Further away from the stationary points of

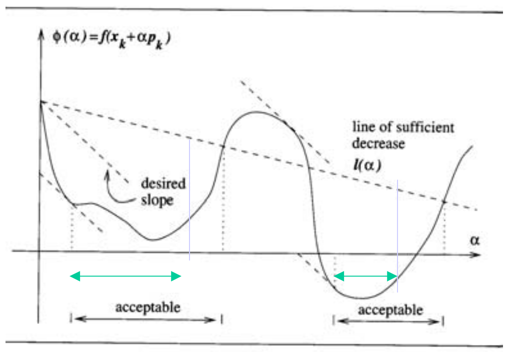
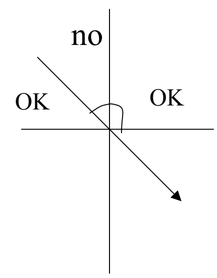
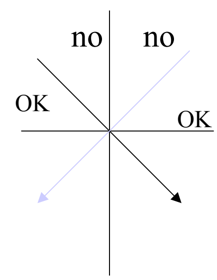


Figure 3.5 Step lengths satisfying the Wolfe conditions.





angle	0	30	45	60	90	120	135	150	180
tan	0	.5774	1	1.732	∞	-1.732	-1	$-.5774$	0
$\lvert \tan \rvert$	0	.577	1	1.732	∞	1.732	1	.5774	0

Goldstein conditions

$$f(x_k) + (1-c)\alpha_k \nabla f_k^T p_k \le f(x_k + \alpha p_k) \le f(x_k) + c\,\alpha_k \nabla f_k^T p_k$$

To control step length from the below $\quad 0 < c < \frac{1}{2}$

Sufficient decrease

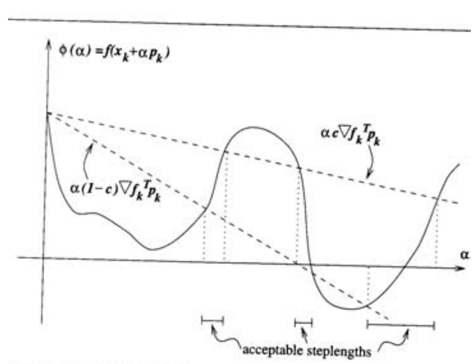


Figure 3.6 The Goldstein conditions.

Disadvantage:
It may exclude minimizers

Goldstein conditions

$$f(x_k) + (1-\mathrm{c})\alpha_\mathrm{k} \nabla f_k^T p_k \le f(x_k + \alpha p_k) \le f(x_k) + c\ \alpha_k \nabla f_k^T p_k$$

To control step length from the below $0 < c < \frac{1}{2}$

(finite difference approximation)

$$\nabla f(x_k + \alpha p_k)^T p_k \ge c_2 \nabla f_k^T(x_k) p_k, \quad c_2 \in (c_1, 1)$$

(Wolf's curvature condition)