Preliminaries

Lecture-2

Eigen Vectors and Eigen Values

The eigen vector, x, of a matrix *A* is a special vector, with the following property

$$Ax = \lambda x$$ Where λ is called eigen value

To find eigen values of a matrix A first find the roots of:

$$\det(A - \lambda I) = 0$$

Then solve the following linear system for each eigen value to find corresponding eigen vector

$$(A - \lambda I)x = 0$$

Example

$$A=\begin{bmatrix}-1 & 2 & 0\\ 0 & 3 & 4\\ 0 & 0 & 7\end{bmatrix}$$

Eigen Values

$$\lambda_1=7,\ \lambda_2=3, \lambda_3=-1$$

$$\mathbf{x_1}=\begin{bmatrix}1\\4\\4\end{bmatrix},\ \mathbf{x_2}=\begin{bmatrix}1\\2\\0\end{bmatrix},\ \mathbf{x_3}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Eigen Vectors

Eigen Values

$$\det(A-\lambda I)=0$$

$$\det(\begin{bmatrix}-1 & 2 & 0\\ 0 & 3 & 4\\ 0 & 0 & 7\end{bmatrix}-\lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix})=0$$

$$\det(\begin{bmatrix}-1-\lambda & 2 & 0\\ 0 & 3-\lambda & 4\\ 0 & 0 & 7-\lambda\end{bmatrix})=0$$

$$(-1-\lambda)((3-\lambda)(7-\lambda)-0)=0$$

$$(-1-\lambda)(3-\lambda)(7-\lambda)=0$$

$$\lambda=-1,\quad \lambda=3,\quad \lambda=7$$

Eigen Vectors

$\lambda = -1 \qquad (A - \lambda I)x = 0$

$$\left(\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 + 2x_2 + 0 = 0$

$0 + 4x_2 + 4x_3 = 0$

$0 + 0 + 8x_3 = 0$

$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Determinant

$$trace(A) = \sum_{i=1}^{n} A_{ii}$$

$$trace(A) = \sum_{i=1}^{n} \lambda_i \text{ where } \lambda_i \text{ are eigen values}$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

$$\det AB = (\det A)(\det B)$$

$$\det A^{-1} = \frac{1}{\det A}$$

$\det A = 0$ if and only if A is singular

$QQ^T = Q^T Q = I$, Q is orthogonal

$$Q^{-1} = Q^T$$

$$\det Q = \det Q^T = \pm 1$$

Rotation matrices are Orthogonal (orthonormal) Matrices

$$(R_\theta^Z)^{-1} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R_\theta^Z)^{-1} = (R_\theta^Z)^T$$
$$(R_\theta^Z)(R_\theta^Z)^T = I$$

Positive-definite

A symetric $n \times n$ matrix is positive definite if

$X^T AX > 0.$

- All diagonal elements of a positive-definite matrix are strictly positive
- Negative definite matrix has all negative eigenvalues
- If all eigenvalues of a symmetric matrix are non-negative, it is said to be Positive semi-definite
- If a matrix has both positive and negative eigenvalues, it is said to be indefinite

Matrix Factorization

$A = LU$, LU decomposition, L is a Lower triangular, and U is a upper triangular

$A = CU$, QR decompoistion, C is orthonormal, U is upper trinagular matrix

Singular Value Decomposition (SVD)

Theorem: Any m by n matrix A, for which $m \geq n$,can be written as

$$A = O_1 \Sigma O_2$$

mxn mxn nxn nxn

Σ is diagonal

O_1 , O_2 are orthogonal

$O_1^T O_1 = O_2^T \; O_2 = I$

Singular Value Decomposition (SVD)

• For some linear systems Ax=b, Gaussian Elimination or LU decomposition does not work, because matrix A is singular, or very close to singular. SVD will not only diagnose for you, but it will solve it.

Singular Value Decomposition (SVD)

If A is square, then O_1, Σ, O_2 are all square.

$$O_1^{-1} = O_1^T$$

$$O_2^{-1} = O_2^T$$

$$\Sigma^{-1} = diag(\frac{1}{w_j})$$

$$A^{-1} = O_2 diag(\frac{1}{w_j}) O_1$$

Singular Value Decomposition (SVD)

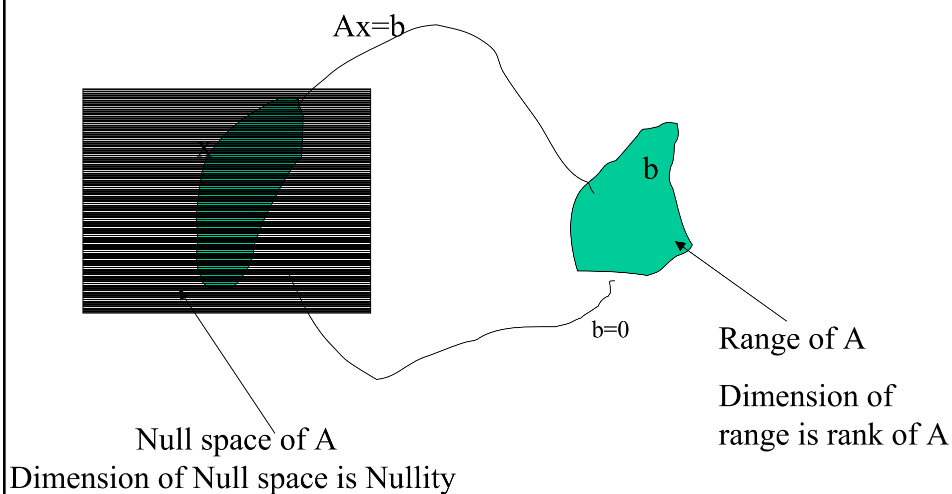
The condition number of a matrix is the ratio of the largest of the w_j to the smallest of w_j . A matrix is singular if the condition number is infinite, it is ill-conditioned if the condition number is too large.

Singular Value Decomposition (SVD)

$$Ax = b$$

- If A is singular, some subspace of "x" maps to zero; the dimension of the null space is called "nullity".
- Subspace of "b" which can be reached by "A" is called range of "A", the dimension of range is called "rank" of A.

Range and Null Space



Singular Value Decomposition (SVD)

- If A is non-singular its rank is “n”.
- If A is singular its rank <n.
- Rank+nullity=n

Singular Value Decomposition (SVD)

$$A = O_1 \Sigma O_2$$

- SVD constructs orthonormal basses of null space and range.
- Columns of O_1 with non-zero w_j spans range.
- Columns of O_2 with zero w_j spans null space.

Solution of Linear System

- How to solve Ax=b, when A is singular?
- If "b" is in the range of "A" then system has many solutions.
- Replace $\frac{1}{w_j}$ by zero if $w_j = 0$

$$x = O_2[diag(\frac{1}{w_j})]O_1^T b$$

Solution of Linear System

If b is not in the range of A, above eq still gives the solution, which is the best possible solution, it minimizes:

$$r \equiv | Ax - b |$$

Cholesky Factorization

A positive-definite symmetric matrix A can be written:

$$A = LDL^T$$

$$A = LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T = \overline{L}\overline{L}^T = R^T R$$

L is unit lower triangular matrix
D is a diagonal matrix with strict
Positive elements
are general lower triangular
and general upper triangular matrices

Spectral Decomposition of A Symmetric Matrix

$$Au_i = \lambda_i u_i$$

$$A = U\Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

Norms

$$\| X \|_1 = \sum_{i=1}^{n} | x_i |, \text{ vector norm}$$

$$|\| X \|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = (X^T X)^{\frac{1}{2}}, \text{ vector norm}$$

$$\| \mathbf{A} \| = \max_{\|x\|=1} \|\mathbf{A}X\|, \text{ matrix norm}$$

$$\| \mathbf{A} \|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Condition Number

Condition number

$k(A) = \| A \| \| A^{-1} \|$

The matrix A is well-conidtioned if K(A) is close to one and is not well-conditioned, when K(A) is significantly greater than one.

1-D Functions

Finding the zero of a function

Bisection Method

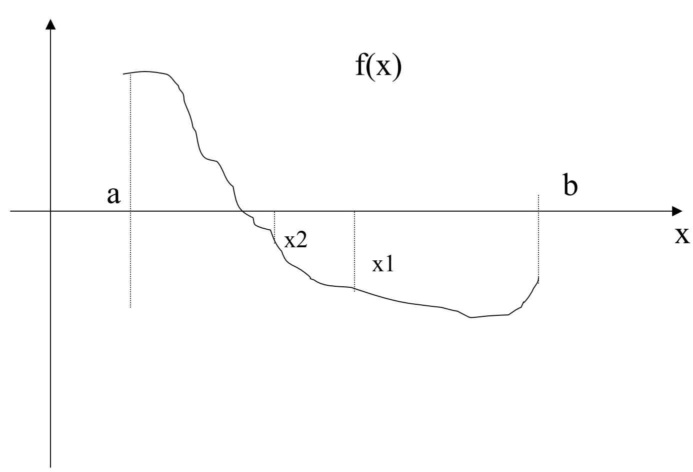
•Find a solution to f(x)=0 on the interval [a,b], where f(a)<0 and f(b)>0 have opposite signs.

–Compute the mid point,m, of [a,b], if f(m)=0, then done

– else if f(m)>0, then b=m, else a=m

$$|p_n - p| \le \frac{b-a}{2^n}$$

Bisection Method



Newton's Method

Suppose that the function f is twice continously differentiable on the interval $[a,b]$; that is $f \in C^2[a,b]$.

$f'(\bar{x}) \neq 0, |\bar{x} - p|$ is small. Taylor series around $\bar{x}$

$$f(x) = f(\bar{x}) + (x-\bar{x})f'(\bar{x}) + \frac{(x-\bar{x})^2}{2} f''(\xi(p)),$$

$\xi(x)$ lies between x and $\bar{x}$.

$$f(p) = 0 \approx f(\bar{x}) + (p-\bar{x})f'(\bar{x}) \quad |\bar{x} - p| \text{ is small.}$$

$$p = \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}$$

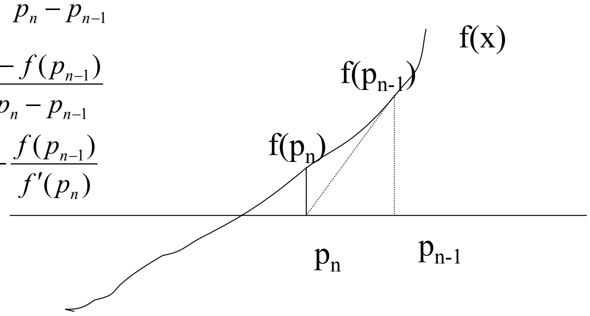
$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Newton's Method

$$f'(p_n) = \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$$

$$f'(p_n) = \frac{0 - f(p_{n-1})}{p_n - p_{n-1}}$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)}$$



Zero of the tangent line

Secant Method

If derivative can not be computed
Use finite difference approximation

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)}$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Theorem

Let $f \in C^2[a,b]$. If $p \in [a,b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists $\delta > 0$ such that Netwon's method generates a sequence p_n converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.