Lecture-17

Theorems 5.8 and 5.9

&

Levenberg-Marquadet

Convergence of Algorithms with restarts

We can use Theorem 5.7 to prove global convergence for algorithms, which are periodically started by setting
If restarts occur at

Since at the restarts

$\cos\theta_k = -1$

$$\sum_{k\geq 0} \cos^2\theta_k \|\nabla f_k\|^2 < \infty$$

$$\sum_{k=k_1,k_2,\ldots} \|\nabla f_k\|^2 < \infty$$

If the restarts are done after every *n* iterations, the sequence is infinite

$\lim_{j\to\infty} \|\nabla f_{k_j}\| = 0$

Therefore a subsequence of gradients will approach to zero:

$$\liminf_{k\to\infty} \|\nabla f_k\| = 0$$

Theorem 5.8

Suppose that the function is Lipschitz continuously differentiable, $\|\nabla f_k\| \le \bar{\gamma}$, and Algorithm 5.4 is implemented with a line search that satisfies strong Wolfe conditions, with $0<c_2<1/2$. Then

$$\liminf_{k\to\infty} \|\nabla f_k\| = 0$$

Proof

$$\liminf_{k\to\infty} \|\nabla f_k\| = 0$$

Proof by contradiction:
Assume that:

$$\|\nabla f_k\| \ge \gamma \quad \text{for all } k \text{ sufficiently large}, \gamma > 0$$

$$\chi_1 \frac{\|\nabla f_k\|}{\|p_k\|} \le \cos\theta_k \le \chi_2 \frac{\|\nabla f_k\|}{\|p_k\|}, \ \forall k = 0,1\ldots$$

We have proved this (Lecture 16)

Now $\sum_{k\ge 0} \cos^2\theta_k \|\nabla f_k\|^2 < \infty$ ⇒ $\sum_{k=1}^{\infty} \frac{\|\nabla f_k\|^4}{\|p_k\|^2} < \infty$ (D)

Proof

$$p_{k+1} \leftarrow -\nabla f_{k+1} + \beta_{k+1}^{FR} p_k;$$
$$p_k = -\nabla f_k + \beta_k^{FR} p_{k-1}$$

$$p_k^T p_k = (-\nabla f_k + \beta_k^{FR} p_{k-1})^T (-\nabla f_k + \beta_k^{FR} p_{k-1})$$
$$p_k^T p_k = \nabla f_k^T \nabla f_k - \beta_k^{FR} \nabla f_k^T p_{k-1} - \beta_k^{FR} p_{k-1}^T \nabla f_k + (\beta_k^{FR})^2 p_{k-1}^T p_{k-1}$$
$$p_k^T p_k = \nabla f_k^T \nabla f_k - 2\beta_k^{FR} \nabla f_k^T p_{k-1} + (\beta_k^{FR})^2 p_{k-1}^T p_{k-1}$$

$$\| p_k \|^2 \le \| \nabla f_k \|^2 + 2\beta_k^{FR} | \nabla f_k^T p_{k-1} | + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$

From (C)

$$| \nabla f_k^T p_{k-1} | \le -c_2 \nabla f_{k-1}^T p_{k-1} \le \frac{c_2}{1-c_2} \| \nabla f_{k-1} \|^2$$

$$\le \| \nabla f_k \|^2 + \frac{2c_2}{1-c_2} \beta_k^{FR} \| \nabla f_{k-1} \|^2 + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$
$$= \| \nabla f_k \|^2 + \frac{2c_2}{1-c_2} \frac{\nabla f_k^T \nabla f_k}{\nabla f_{k-1}^T \nabla f_{k-1}} \| \nabla f_{k-1} \|^2 + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$

$$\beta_{k+1}^{FR} = \frac{\nabla f_{k+1}^T \nabla f_{k+1}}{\nabla f_k^T \nabla f_k}$$

$$\| p_k \|^2 \le \left(\frac{1+c_2}{1-c_2} \right) \| \nabla f_k \|^2 + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$

Proof

$$| \nabla f_{k+1}^T p_k | \le -c_2 \nabla f_k^T p_k$$ Wolf's condition

$$| \nabla f_k^T p_{k-1} | \le -c_2 \nabla f_{k-1}^T p_{k-1} \qquad \text{(A)}$$

$$-\frac{1}{1-c_2} \le \frac{\nabla f_k^T p_k}{\| \nabla f_k \|^2} \le \frac{2c_2 - 1}{1-c_2}, \ \forall k = 0,1\ldots$$ Lemma 5.6

$$-\frac{1}{1-c_2} \le \frac{\nabla f_{k-1}^T p_{k-1}}{\| \nabla f_{k-1} \|^2} \le \frac{2c_2 - 1}{1-c_2} \qquad \text{(B)}$$

Combining (A) and (B)

$$| \nabla f_k^T p_{k-1} | \le -c_2 \nabla f_{k-1}^T p_{k-1} \le \frac{c_2}{1-c_2} \| \nabla f_{k-1} \|^2 \qquad \text{(C)}$$

Proof

$$\| p_k \|^2 \le \left(\frac{1+c_2}{1-c_2} \right) \| \nabla f_k \|^2 + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$

$$c_3 = (1+c_2)/(1-c_2) \ge 1$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^2 + (\beta_k^{FR})^2 \| p_{k-1} \|^2$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^2 + (\beta_k^{FR})^2 [c_3 \| \nabla f_{k-1} \|^2 + (\beta_{k-1}^{FR})^2 \| p_{k-2} \|^2]$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^2 + (\beta_k^{FR})^2 c_3 \| \nabla f_{k-1} \|^2 + (\beta_k^{FR})^2 (\beta_{k-1}^{FR})^2 \| p_{k-2} \|^2$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^2 + \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-1} \|^4} c_3 \| \nabla f_{k-1} \|^2 + \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-1} \|^4} \frac{\| \nabla f_{k-1} \|^4}{\| \nabla f_{k-2} \|^4} \| p_{k-2} \|^2$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^2 + \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-1} \|^2} c_3 + \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-2} \|^4} \| p_{k-2} \|^2$$

$$\beta_{k+1}^{FR} = \frac{\nabla f_{k+1}^T \nabla f_{k+1}}{\nabla f_k^T \nabla f_k}$$

$$= c_3 \| \nabla f_k \|^4 \left[\frac{1}{\| \nabla f_k \|^2} + \frac{1}{\| \nabla f_{k-1} \|^2} \right] + \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-2} \|^4} \| p_{k-2} \|^2$$

$$(\beta_k^{FR})^2 (\beta_{k-1}^{FR})^2 \cdots (\beta_{k-i}^{FR})^2 = \frac{\| \nabla f_k \|^4}{\| \nabla f_{k-i-1} \|^4}$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^4 \sum_{j=1}^{k} \| \nabla f_j \|^{-2}$$

Proof

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^4 \sum_{j=1}^{k} \| \nabla f_j \|^{-2}$$

$$\| p_k \|^2 \le c_3 \| \nabla f_k \|^4 \sum_{j=1}^{k} \frac{1}{\| \nabla f_j \|^2}$$

$$\| p_k \|^2 \le c_3 \bar{\gamma}^4 k \frac{1}{\gamma^2}$$

$$\| \nabla f_k \| \le \bar{\gamma} \quad \| \nabla f_k \| \ge \gamma$$

$$\sum_{k=1}^{\infty} \frac{1}{\| p_k \|^2} \ge \gamma_4 \sum_{k=1}^{\infty} \frac{1}{k}$$

Proof

$$\sum_{k=1}^{\infty} \frac{1}{\| p_k \|^2} \geq \gamma_4 \sum_{k=1}^{\infty} \frac{1}{k}$$

Then from (D)

$$\sum_{k=1}^{\infty} \frac{\| \nabla f_k \|^4}{\| p_k \|^2} < \infty$$

Assume $\| \nabla f_k \| \geq \gamma$ for all k sufficiently large

$\Rightarrow$ $\sum_{k=1}^{\infty} \frac{1}{\| p_k \|^2} < \infty$ $\qquad$ $\gamma_4 \sum_{k=1}^{\infty} \frac{1}{k} \leq \sum_{k=1}^{\infty} \frac{1}{\| p_k \|^2}$

$$\gamma_4 \sum_{k=1}^{\infty} \frac{1}{k} \leq \sum_{k=1}^{\infty} \frac{1}{\| p_k \|^2} < \infty \quad \Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k} < \infty$$

Which is not true

General Result

In general, if we can show that there is a constant c_4 such that:

$$\cos \theta_k \geq c_4 \frac{\| \nabla f_k \|}{\| p_k \|} \quad k = 1, 2 \ldots, c_4 > 0$$

And another constant c_5 such that:

$$\frac{\| \nabla f_k \|}{\| p_k \|} \geq c_5 > 0 \quad k = 1, 2 \ldots$$

Then using Theorem 5.7

$$\sum_{k \geq 0} \cos^2 \theta_k \| \nabla f_k \|^2 < \infty$$

$$\sum_{k \geq 0} {c_4}^2 \frac{\| \nabla f_k \|^2}{\| p_k \|^2} \| \nabla f_k \|^2 < \infty$$

$$\sum_{k \geq 0} {c_4}^2 \| \nabla f_k \|^2 \, {c_5}^2 \frac{1}{\| \nabla f_k \|^2} \| \nabla f_k \|^2 < \infty$$

We can show:

$$\lim_{k \to \infty} \| \nabla f_k \| = 0$$

General Result

This result can be established for PR assuming the function is strongly convex and then an exact line search is used.

For general functions, it is not possible to prove this for PR.

Theorem 5.9

Consider PR-GC with an ideal line search. There exists a twice continuously differentiable function *f*: $\boldsymbol{R}^3 \rightarrow \boldsymbol{R}$ and a starting point x_0, such that the sequence of gradients $\{\|\nabla f_k\|\}$ is bounded away from zero.

Proof of this requires that the consecutive search directions become almost negatives of each others. In the ideal line search this can only happen when $\beta_k < 0$ so that suggest restart of PR.

$$\beta_{k+1}^{+} = \max(\beta_{k+1}^{PR}, 0)$$

Lecture-18

Levenberg-Marquadet

Weighted Non-linear least squares fit

Consider a set of non-linear equations:

$$y_i = f(x_i, a)$$

Our aim is to determine a vector such that the following is minimized:

$$\Psi(a) = \sum_{i=1}^{N} \left(\frac{y_i - f(x_i, a)}{\sigma_i} \right)^2$$

weights

Function to be minimized $$\Psi(a) = \sum_{i=1}^{N} \left(\frac{y_i - f(x_i, a)}{\sigma_i} \right)^2 \qquad \text{(D)}$$

Newton's (Inverse Hessian) method:

$$a_{next} = a_{current} + D^{-1}[-\nabla\Psi(a_{current})] \qquad \text{(A)}$$

Where D is a Hessian matrix

The gradient is given by:

$$\frac{\partial\Psi}{\partial a_k} = -2\sum_i \left[\frac{y_i - f(x_i, a)}{\sigma_i^2} \right] \frac{\partial f(x_i, a)}{\partial a_k} \quad k = 1, \ldots, m$$

The Hessian is given by:

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left[\frac{\partial f(x_i, y)}{\partial a_k} \frac{\partial f(x_i, y)}{\partial a_l} - [y_i - f(x_i, a)] \frac{\partial^2 f(x_i, a)}{\partial a_k \partial a_l} \right] \qquad \text{(B)}$$

Let us define:

$$\beta_k \equiv -\frac{1}{2}\frac{\partial\Psi}{\partial \mathrm{a_k}} \qquad \alpha_{kl} \equiv \frac{1}{2}\frac{\partial^2\Psi}{\partial \mathrm{a_k} \partial a_l}$$

Now the Hessian is given by:

$$[\alpha] = \frac{1}{2} D$$

Newton's method (A) can be written as

$$a_{next} = a_{current} + D^{-1}[-\nabla\Psi(a_{current})] \qquad \text{(A)}$$

$$\sum_{l=1}^{M} \alpha_{kl} \delta a_l = \beta_k \qquad \text{(E)}$$ Where $$\delta a = a_{next} - a_{current}$$

The gradient descent is given by:

$$a_{next} = a_{current} + const[-\nabla\Psi(a_{current})]$$

$$\delta a_l = const\ \beta_l \quad \text{(C)} \qquad \delta a = a_{next} - a_{current} \qquad \beta_k \equiv -\frac{1}{2}\frac{\partial\Psi}{\partial \mathrm{a_k}}$$

Assume the second term in (B) is zero:

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l} - [y_i - f(x_i,a)]\frac{\partial^2 f(x_i,a)}{\partial a_k \partial a_l}\right] \qquad \text{(B)}$$

$$\frac{\partial^2\Psi}{\partial a_k \partial a_l} = 2\sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l}\right]$$

Now $$\alpha_{kl} = \sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l}\right]$$

$$\beta_k \equiv -\frac{1}{2}\frac{\partial\Psi}{\partial \mathrm{a_k}} \qquad \alpha_{kl} = \sum_{i=1}^{N}\frac{1}{\sigma_i^2}\left[\frac{\partial f(x_i,a)}{\partial a_k}\frac{\partial f(x_i,a)}{\partial a_l}\right]$$

$$\delta a_l = const\ \beta_l \qquad \text{(C)}$$ Gradient descent

Let the constant be given by

$$const = \frac{1}{\lambda\alpha_{ll}} \qquad \delta a_l = \frac{1}{\lambda\alpha_{ll}}\beta_l \qquad \text{(G)}$$

Now define:

$$\alpha'_{jj} \equiv \alpha_{jj}(1+\lambda) \quad \text{for } i = j$$
$$\alpha'_{kj} \equiv \alpha_{kj} \quad \text{when j} \neq \text{k} \qquad \text{(F)}$$

Newton from (E) $$\sum_{l=1}^{M}\alpha_{kl}\delta a_l = \beta_k$$

Combining (E) and (G) and using (F) $$\sum_{l=1}^{m}\alpha'_{kl}\delta a_l = \beta_k \qquad \text{(H)}$$

Algorithm

1.Start with some initial estimate of *a.*

2. Compute $\Psi(x_i, a)$ (equation D). $\Psi(a) = \sum_{i=1}^{N} \left(\frac{y_i - f(x_i, a)}{\sigma_i} \right)^2$

3. Pick a modest value of λ=.001.

4. Solve linear system (H) for δa and evaluate $\Psi(x_i, a+\delta a)$. $\sum_{l=1}^{m} \alpha'_{kl} \delta a_l = \beta_k$

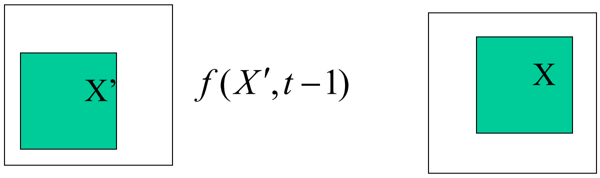
5. If $\Psi(x_i, a+\delta a) \geq \Psi(x_i, a)$, increase λ by a factor of 10, and go to step (4)

6. If $\Psi(x_i, a+\delta a) \leq \Psi(x_i, a)$ decrease λ by a factor of 10, update the trial solution: $a \leftarrow a+\delta a$, and go back to step 4.

Szeliski

Projective

Projective



$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

min

Compute

$$\mathbf{m} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & c_1 & c_2 \end{bmatrix}^T$$

Szeliski (Levenberg-Marquadet)

Hessian $\alpha_{kl} = \sum \frac{\partial e}{\partial m_k} \frac{\partial e}{\partial m_l}$

$b_k = -\sum e \frac{\partial e}{\partial m_k}$ **gradient**

$$\Delta m = (A + \lambda I)^{-1} b$$

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_1}$$

$$E = \sum [f(x', y') - f(x, y)]^2 = \sum e^2$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

Hessian

$$A = \begin{bmatrix}
\sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial a_1}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial a_2}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial a_3}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial a_4}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial b_2}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial b_2}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial b_2}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial b_1} & \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial b_1}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial^2 e}{\partial b_1 \partial c_2} \\
\sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial c_1}\frac{\partial e}{\partial c_2} \\
\sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial a_1} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial a_2} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial a_3} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial a_4} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial b_1} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial b_2} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial c_1} & \sum_x \frac{\partial e}{\partial c_2}\frac{\partial e}{\partial c_2}
\end{bmatrix}$$

$$b = \begin{bmatrix} -\sum_x e \frac{\partial e}{\partial a_1} \\ -\sum_x e \frac{\partial e}{\partial a_2} \\ -\sum_x e \frac{\partial e}{\partial a_3} \\ -\sum_x e \frac{\partial e}{\partial a_4} \\ -\sum_x e \frac{\partial e}{\partial b_1} \\ -\sum_x e \frac{\partial e}{\partial b_2} \\ -\sum_x e \frac{\partial e}{\partial c_1} \\ -\sum_x e \frac{\partial e}{\partial c_2} \end{bmatrix}$$

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}, \quad y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_1} = \frac{x}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_1} = 0$$

$$\frac{\partial x'}{\partial a_2} = \frac{y}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial a_2} = 0$$

$$\frac{\partial x'}{\partial a_3} = 0 \qquad \frac{\partial y'}{\partial a_3} = \frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial a_4} = 0 \qquad \frac{\partial y'}{\partial a_4} = \frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial b_1} = \frac{1}{c_1 x + c_2 y + 1} \qquad \frac{\partial y'}{\partial b_1} = 0$$

$$\frac{\partial x'}{\partial b_2} = 0 \qquad \frac{\partial y'}{\partial b_2} = \frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial x'}{\partial c_1} = \frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_1} = \frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial x'}{\partial c_2} = \frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} \qquad \frac{\partial y'}{\partial c_2} = \frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial e}{\partial a_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_1} = f_{x'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_2} = f_{x'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_3} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_3} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_3} = f_{y'}\frac{x}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial a_4} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial a_4} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial a_4} = f_{y'}\frac{y}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_1} = f_{x'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial b_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial b_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial b_2} = f_{y'}\frac{1}{c_1 x + c_2 y + 1}$$

$$\frac{\partial e}{\partial c_1} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_1} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_1} = f_{x'}\frac{-x(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-x(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\frac{\partial e}{\partial c_2} = \frac{\partial e}{\partial x'}\frac{\partial x'}{\partial c_2} + \frac{\partial e}{\partial y'}\frac{\partial y'}{\partial c_2} = f_{x'}\frac{-y(a_1 x + a_2 y + b_1)}{(c_1 x + c_2 y + 1)^2} + f_{y'}\frac{-y(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}$$

$$\mathbf{b} = \begin{bmatrix} -\sum e f_{x'}\frac{x}{c_1 x + c_2 y + 1} \\ -\sum e f_{x'}\frac{y}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'}\frac{x}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'}\frac{y}{c_1 x + c_2 y + 1} \\ -\sum e f_{x'}\frac{1}{c_1 x + c_2 y + 1} \\ -\sum e f_{y'}\frac{1}{c_1 x + c_2 y + 1} \\ \sum ex\left[\frac{f_{x'}(a_1 x + a_2 y + b_1) + f_{y'}(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}\right] \\ \sum ey\left[\frac{f_{x'}(a_1 x + a_2 y + b_1) + f_{y'}(a_3 x + a_4 y + b_2)}{(c_1 x + c_2 y + 1)^2}\right] \end{bmatrix}$$

Szeliski (Levenberg-Marquadet)

- Start with some initial value of m, and λ=.001

- For each pixel *i* at

- Compute using projective transform.

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$

$$y' = \frac{a_3 x + a_4 y + b_2}{c_1 x + c_2 y + 1}$$

- Compute

- Compute

Szeliski (Levenberg-Marquadet)

-Compute and

-Solve system

$$(A - \lambda I)\Delta m = b$$

-Update

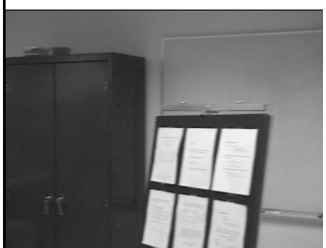
$$m^{t+1} = m^t + \Delta m$$

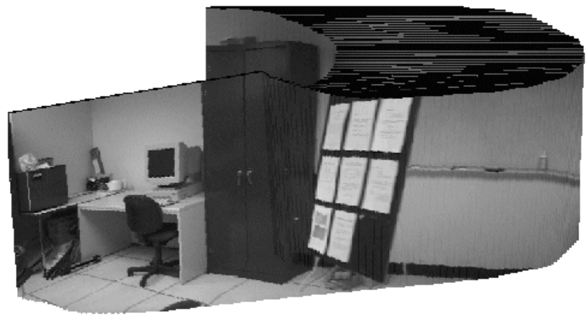
Szeliski (Levenberg-Marquadet)

- check if error has decreased, if not increase by a factor of 10 and compute a new Δm
- If error has decreased, decrease by a factor of 10 and compute a new Δm
- Continue iteration until error is below threshold.

Video Mosaic

Video Mosaic





Video Mosaic