Eigen Vectors and Eigen Values

The eigen vector, $\mathbf{x}$, of a matrix $A$ is a special vector, with the following property

$$A\mathbf{x} = \lambda \mathbf{x}$$

Where $\lambda$ is called eigen value

To find eigen values of a matrix $A$ first find the roots of:

$$\det(A - \lambda I) = 0$$

Then solve the following linear system for each eigen value to find corresponding eigen vector

$$(A - \lambda I)\mathbf{x} = 0$$
Example

\[ A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} \]

Eigen Values

\[ \lambda_1 = 7, \lambda_2 = 3, \lambda_3 = -1 \]

\[ x_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Eigen Vectors

Eigen Values

\[ \det(A - \lambda I) = 0 \]

\[ \det(\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}) = 0 \]

\[ \det(\begin{bmatrix} -1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 7-\lambda \end{bmatrix}) = 0 \]

\[ (-1-\lambda)(3-\lambda)(7-\lambda) - 0 = 0 \]

\[ (-1-\lambda)(3-\lambda)(7-\lambda) = 0 \]

\[ \lambda = -1, \quad \lambda = 3, \quad \lambda = 7 \]
Eigen Vectors

\[ \lambda = -1 \quad (A - \lambda I)x = 0 \]

\[
\begin{bmatrix}
-1 & 2 & 0 \\
0 & 3 & 4 \\
0 & 0 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 2 & 0 \\
0 & 4 & 4 \\
0 & 0 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[ 0 + 2x_2 + 0 = 0 \]
\[ 0 + 4x_2 + 4x_3 = 0 \]
\[ 0 + 0 + 8x_3 = 0 \]

\[ x_1 = 1, \quad x_2 = 0, \quad x_3 = 0 \]

\[ x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ trace(A) = \sum_{i=1}^{n} A_{ii} \]

\[ trace(A) = \sum_{i=1}^{n} \lambda_i \text{ where } \lambda_i \text{ are eigen values} \]

\[ \det(A) = \prod_{i=1}^{n} \lambda_i \]

\[ \det A = 0 \text{ if and only if } A \text{ is singular} \]

\[ \det AB = (\det A)(\det B) \]

\[ \det A^{-1} = \frac{1}{\det A} \]

\[ QQ^T = Q^TQ = I, \text{ } Q \text{ is orthogonal} \]

\[ Q^{-1} = Q^T \]

\[ \det Q = \det Q^T = \pm 1 \]
Rotation matrices are Orthogonal (orthonormal) Matrices

\[
(R_\theta^T)^T = \begin{bmatrix}
\cos \Theta & \sin \Theta & 0 \\
-\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \Theta & \sin \Theta & 0 \\
-\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
(R_\theta^T)(R_\theta^T)^T = I
\]

Positive-definite

A symmetric n×n matrix is positive definite if 
\[X^TAX > 0.\]

- If all eigenvalues of a symmetric matrix are non-negative, it is said to be Positive semi-definite
- If all eigenvalues of a symmetric matrix are non-negative, it is said to be Positive semi-definite
- If a matrix has both positive and negative eigenvalues, it is said to be indefinite
Matrix Factorization

$A = LU$, LU decomposition, $L$ is a Lower triangular, and $U$ is a upper triangular

$A = CU$, QR decomposition, $C$ is orthonormal, $U$ is upper triangular matrix

Singular Value Decomposition (SVD)

Theorem: Any $m$ by $n$ matrix $A$, for which $m \geq n$, can be written as

$$A = O_1 \Sigma O_2^T$$

$\Sigma$ is diagonal

$O_1, O_2$ are orthogonal

$O_1^TO_1 = O_2^TO_2 = I$
Singular Value Decomposition (SVD)

- For some linear systems $Ax=b$, Gaussian Elimination or LU decomposition does not work, because matrix $A$ is singular, or very close to singular. SVD will not only diagnose for you, but it will solve it.

If $A$ is square, then $O_1, \Sigma, O_2$ are all square.

$$O_1^{-1} = O_1^T$$
$$O_2^{-1} = O_2^T$$
$$\Sigma^{-1} = diag\left(\frac{1}{w_j}\right)$$
$$A^{-1} = O_2 diag\left(\frac{1}{w_j}\right) O_1$$
Singular Value Decomposition (SVD)

The condition number of a matrix is the ratio of the largest of the $w_j$ to the smallest of $w_j$. A matrix is singular if the condition number is infinite, it is ill-conditioned if the condition number is too large.

Singular Value Decomposition (SVD)

$$Ax = b$$

- If $A$ is singular, some subspace of “$x$” maps to zero; the dimension of the null space is called “nullity”.
- Subspace of “$b$” which can be reached by “$A$” is called range of “$A$”, the dimension of range is called “rank” of $A$. 

Singular Value Decomposition (SVD)

- If A is non-singular its rank is “n”.
- If A is singular its rank <n.
- Rank+nullity=n

Singular Value Decomposition (SVD)

- SVD constructs orthonormal basses of null space and range.
- Columns of $O_1$ with non-zero $w_j$ spans range.
- Columns of $O_2$ with zero $w_j$ spans null space.
Solution of Linear System

• How to solve $Ax=b$, when $A$ is singular?

• If “$b$” is in the range of “$A$” then system has many solutions.

• Replace $\frac{1}{w_j}$ by zero if $w_j = 0$

$$x = O_2[\text{diag}(\frac{1}{w_j})]O_1^Tb$$

Solution of Linear System

If $b$ is not in the range of $A$, above eq still gives the solution, which is the best possible solution, it minimizes:

$$r \equiv |Ax - b|$$
Cholesky Factorization

A positive-definite symmetric matrix $A$ can be written:

$$A = LDL^T$$

$$A = LD^2D^2L^T = LL^T = RR^T$$

$L$ is unit lower triangular matrix
$D$ is a diagonal matrix with strict Positive elements
$L, R$ are general lower triangular and general upper triangular matrices

Spectral Decomposition of A Symmetric Matrix

$$Au_i = \lambda_i u_i$$

$$A = UAUA^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$
Norms

\[ \| X \|_1 = \sum_{i=1}^{n} |x_i|, \text{ vector norm} \]
\[ \| X \|_2 = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} = (X^T X)^{\frac{1}{2}}, \text{ vector norm} \]
\[ \| A \| = \max_{\| a \|=1} \| A X \|, \text{ matrix norm} \]
\[ \| A \|_\infty = \max \sum_{j=1}^{n} |a_{ij}| \quad \text{for} \quad 1 \leq i \leq n \]

Norms

Condition number
\[ k(A) = \| A \| \| A^{-1} \| \]

The matrix A is well-conditioned if K(A) is close to one and is not well-conditioned, when K(A) is significantly greater than one.
1-D Functions

Finding the zero of a function

Bisection Method

• Find a solution to $f(x)=0$ on the interval $[a,b]$, where $f(a)<0$ and $f(b)>0$ have opposite signs.
  – Compute the mid point, $m$, of $[a,b]$, if $f(m)=0$, then done
  – else if $f(m)>0$, then $b=m$, else $a=m$

$$|p_n - p| \leq \frac{b-a}{2^n}$$
Bisection Method

Newton’s Method

Suppose that the function $f$ is twice differentiable on the interval $[a, b]$; that is $f \in C^2[a, b]$. $f'(\bar{x}) \neq 0$, $|\bar{x} - p|$ is small. Taylor series around $\bar{x}$

$$f(x) = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + \frac{(x - \bar{x})^2}{2} f''(\xi(x)),$$

where $\xi(x)$ lies between $x$ and $\bar{x}$.

$$0 = f(\bar{x}) + (p - \bar{x})f'(\bar{x}) \quad |\bar{x} - p| \text{ is small.}$$

$$p = \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$
Newtons’ Method

\[ f'(p_n) = \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}} \]

\[ f'(p_n) = \frac{0 - f(p_{n-1})}{p_n - p_{n-1}} \]

\[ p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)} \]

Secent Method

If derivative can not be computed
Use finite difference approximation

\[ p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \]
Theorem

Let $f \in C^1[a,b]$. If $p \in [a,b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists $\delta > 0$ such that Newton’s method generates a sequence $p_n$ converging to $p$ for any initial approximation $p_0 \in [p - \delta, p + \delta]$. 