$\mathbf{X}_0 = (\mathrm{X}_0, \mathrm{Y}_0, \mathrm{Z}_0)^\top$: 3-D coordinates of a point at time t_0

$\mathbf{T} = (\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3)^\top$: Translation vector

spherical coordinates: slant, θ_T, and tilt, ϕ_T:

$$\mathbf{T} = (\mathrm{sin}\theta_\mathrm{T}\mathrm{cos}\phi_\mathrm{T}, \mathrm{sin}\theta_\mathrm{T}\mathrm{sin}\phi_\mathrm{T}, \mathrm{cos}\theta_\mathrm{T})^\top. \tag{1}$$

$\mathbf{q} = (\mathrm{q}_0, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3)^\top$: Quaternion

$\mathbf{q} = (\sin\frac{\alpha}{2}\mathrm{n}, \cos\frac{\alpha}{2})^\top$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 + q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{2}$$

$$\mathbf{q}(\mathrm{t}) = \begin{pmatrix} \frac{w_x}{w}\sin\frac{wt}{2}, & \frac{w_y}{w}\sin\frac{wt}{2}, & \frac{w_z}{w}\sin\frac{wt}{2}, & \cos\frac{wt}{2} \end{pmatrix}^\top, \tag{3}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

3-D motion:

$$\mathbf{X}_{\mathrm{t}} = \mathbf{R}(\mathbf{q}(\mathrm{t}))\mathbf{X}_0 + \mathrm{t}\mathbf{T}. \tag{4}$$

Perspective projection:

$\mathbf{X} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)$,

$x = f\frac{X_1}{X_3}$, $y = f\frac{X_2}{X_3}$

Unknowns: $\mathbf{a} = (\theta_{\mathrm{T}}, \phi_{\mathrm{T}}, \mathrm{w_x}, \mathrm{w_y}, \mathrm{w_z}, \mathrm{Z}_0)^\top$.

Problem: determine $\mathbf{a}$, given $\{\mathbf{x}_{\mathrm{s}}, \mathbf{x}_{\mathrm{s+1}}, \cdots, \mathbf{x}_{\mathrm{s+N}}\}$,

where $\mathbf{x}_{\mathrm{t}} = (\mathrm{x_t}, \mathrm{y_t})^\top, \mathrm{t} = \mathrm{s}, \cdots, \mathrm{s} + \mathrm{N}$, $s \geq 0$

Let $\mathbf{h}_{\mathrm{t}}(\mathbf{a})$ be the 3-D coordinates of a point at time t, **given parameter a**:

$$\mathbf{h}_{\mathrm{t}}(\mathbf{a}) = \mathbf{R}(\mathbf{q}(\mathrm{t}))\mathbf{X}_0 + \mathrm{t}\mathbf{T}. \tag{5}$$

Given image coordinates $\mathbf{x}_{\mathrm{t}}$, and depth Z_t, its corresponding 3-D coordinates are:

$\mathbf{X}_{\mathrm{t}} = (\mathrm{Z_t x_t/f}, \mathrm{Z_t y_t/f}, \mathrm{Z_t})^\top$.

Let $\mathbf{p} = (0,\ 0,\ 1)$, then $Z_t = \mathbf{p}\ \mathbf{h}_{\mathrm{t}}(\mathbf{a})$. Now, $\mathbf{X}_{\mathrm{t}}$ becomes

$$\mathbf{X}_{\mathrm{t}} = \mathbf{p}\ \mathbf{h}_{\mathrm{t}}(\mathbf{a}) \begin{pmatrix} x_t/f \\ y_t/f \\ 1 \end{pmatrix}.$$

The objective function to be minimized is given by:

$$E^2(\mathbf{a}) = \sum_{\mathrm{t=s}}^{\mathrm{s+N}} \parallel \mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a}) \parallel^2, \tag{6}$$

$$\mathbf{X}_{\mathrm{t}} = \mathbf{p}\ \mathbf{h}_{\mathrm{t}}(\mathbf{a}) \begin{pmatrix} x_t/f \\ \\ y_t/f \\ \\ 1 \end{pmatrix}.$$

$$\mathbf{h}_{\mathrm{t}}(\mathbf{a}) = \mathbf{R}(\mathbf{q}(\mathrm{t}))\mathbf{X}_0 + \mathrm{t}\mathbf{T}. \tag{7}$$

$$(\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})) = (\mathbf{p}\ \mathbf{h}_{\mathrm{t}}(\mathbf{a}) \begin{pmatrix} x_t/f \\ \\ y_t/f \\ \\ 1 \end{pmatrix} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})) \tag{8}$$

$$\frac{\partial}{\partial \mathbf{a}_{\mathrm{i}}}(\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})) = \mathrm{t}\mathbf{p}\ \frac{\partial \mathbf{T}}{\partial \mathbf{a}_{\mathrm{i}}} \begin{pmatrix} x_t/f \\ \\ y_t/f \\ \\ 1 \end{pmatrix} - \mathrm{t}\frac{\partial \mathbf{T}}{\partial \mathbf{a}_{\mathrm{i}}}, \quad \mathrm{i} = 1, 2, \tag{9}$$

$$\frac{\partial}{\partial \mathbf{a}_{\mathrm{i}}}(\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})) = \mathbf{p}\ \frac{\partial \mathbf{R}}{\partial \mathbf{a}_{\mathrm{i}}}\ \mathbf{X}_0 \begin{pmatrix} x_t/f \\ \\ y_t/f \\ \\ 1 \end{pmatrix} - \frac{\partial \mathbf{R}}{\partial \mathbf{a}_{\mathrm{i}}}\ \mathbf{X}_0, \quad \mathrm{i} = 3, 4, 5, \tag{10}$$

$$\frac{\partial}{\partial \mathbf{a}_6}(\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})) = \left\{ \mathbf{pR} \begin{pmatrix} x_0/f \\ \\ y_0/f \\ \\ 1 \end{pmatrix} \right\} \begin{pmatrix} x_t/f \\ \\ y_t/f \\ \\ 1 \end{pmatrix} - \mathbf{R} \begin{pmatrix} x_0/f \\ \\ y_0/f \\ \\ 1 \end{pmatrix}, \tag{11}$$

$$\mathbf{T} = (\mathrm{sin}\theta_{\mathrm{T}}\mathrm{cos}\phi_{\mathrm{T}}, \mathrm{sin}\theta_{\mathrm{T}}\mathrm{sin}\phi_{\mathrm{T}}, \mathrm{cos}\theta_{\mathrm{T}})^{\top}. \tag{12}$$

where

$$\frac{\partial \mathbf{T}}{\partial \mathbf{a}_1} = \frac{\partial \mathbf{T}}{\partial \theta_T} = \begin{pmatrix} \cos\theta_T \ \ \cos\phi_T \\ \\ \cos\theta_T \ \ \sin\phi_T \\ \\ -\sin\theta_T \end{pmatrix}, \qquad \frac{\partial \mathbf{T}}{\partial \mathbf{a}_2} = \frac{\partial \mathbf{T}}{\partial \phi_T} = \begin{pmatrix} -\sin\theta_T \ \ \sin\phi_T \\ \\ \sin\theta_T \ \ \cos\phi_T \\ \\ 0 \end{pmatrix},$$

$\mathbf{q} = (\mathrm{q}_0, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3)^\top$: Quaternion

$$\mathbf{q}(\mathrm{t}) = \left(\frac{w_x}{w} \sin\frac{wt}{2}, \quad \frac{w_y}{w} \sin\frac{wt}{2}, \quad \frac{w_z}{w} \sin\frac{wt}{2}, \quad \cos\frac{wt}{2} \right)^\top, \tag{13}$$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 + q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{14}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

$$\frac{\partial \mathbf{R}}{\partial \mathbf{a}_3} = \frac{\partial \mathbf{R}}{\partial w_x} = \sum_{k=0}^{3} \frac{\partial \mathbf{R}}{\partial q_k}\frac{\partial q_k}{\partial w_x} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_x}, \quad \frac{\partial \mathbf{R}}{\partial \mathbf{a}_4} = \frac{\partial \mathbf{R}}{\partial w_y} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_y}, \quad \frac{\partial \mathbf{R}}{\partial \mathbf{a}_5} = \frac{\partial \mathbf{R}}{\partial w_z} = \sum_{k=0}^{3} D_k \frac{\partial q_k}{\partial w_z},$$

where $D_k = \partial \mathbf{R}/\partial \mathrm{q_k}$, $k = 0, 1, 2, 3$, are given by

$$\mathbf{D}_0 = 2\begin{pmatrix} q_0 & q_1 & q_2 \\ q_1 & -q_0 & q_3 \\ q_2 & -q_3 & -q_0 \end{pmatrix}, \qquad \mathbf{D}_1 = 2\begin{pmatrix} -q_1 & q_0 & -q_3 \\ q_0 & q_1 & q_2 \\ q_3 & q_2 & -q_1 \end{pmatrix},$$

$$\mathbf{D}_2 = 2\begin{pmatrix} -q_2 & q_3 & q_0 \\ -q_3 & -q_2 & q_1 \\ q_0 & q_1 & q_2 \end{pmatrix}, \qquad \mathbf{D}_3 = 2\begin{pmatrix} q_3 & q_2 & -q_1 \\ -q_2 & q_3 & q_0 \\ q_1 & -q_0 & -q_3 \end{pmatrix}.$$

$\mathbf{q} = (\mathrm{q}_0, \mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3)^\top$: Quaternion

$$\mathbf{q}(\mathrm{t}) = \begin{pmatrix} \frac{w_x}{w}\sin\frac{wt}{2}, & \frac{w_y}{w}\sin\frac{wt}{2}, & \frac{w_z}{w}\sin\frac{wt}{2}, & \cos\frac{wt}{2} \end{pmatrix}^\top, \tag{15}$$

Rotation Matrix:

$$\mathbf{R} = \begin{pmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 + q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{pmatrix}, \tag{16}$$

$w = \sqrt{w_x^2 + w_y^2 + w_z^2}$, and $w \neq 0$.

Now, $\partial q_k / \partial \mathbf{a}_\mathrm{i}$, $k = 0, 1, 2, 3$, $i = 3, 4, 5$, are given by

$\frac{\partial q_0}{\partial w_x} = (\frac{1}{w} - \frac{w_x^2}{w^3})\sin\frac{wt}{2} + \frac{t}{2}\frac{w_x^2}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_1}{\partial w_x} = -\frac{w_x w_y}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_x w_y}{w^2}\cos\frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_x} = -\frac{w_x w_z}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_x w_z}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_3}{\partial w_x} = -\frac{t}{2}\frac{w_x}{w}\sin\frac{wt}{2}$,

$\frac{\partial q_0}{\partial w_y} = -\frac{w_x w_y}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_x w_y}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_1}{\partial w_y} = (\frac{1}{w} - \frac{w_y^2}{w^3})\sin\frac{wt}{2} + \frac{t}{2}\frac{w_y^2}{w^2}\cos\frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_y} = -\frac{w_y w_z}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_y w_z}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_3}{\partial w_y} = -\frac{t}{2}\frac{w_y}{w}\sin\frac{wt}{2}$,

$\frac{\partial q_0}{\partial w_z} = -\frac{w_x w_z}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_x w_z}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_1}{\partial w_z} = -\frac{w_y w_z}{w^3}\sin\frac{wt}{2} + \frac{t}{2}\frac{w_y w_z}{w^2}\cos\frac{wt}{2}$,

$\frac{\partial q_2}{\partial w_z} = (\frac{1}{w} - \frac{w_z^2}{w^3})\sin\frac{wt}{2} + \frac{t}{2}\frac{w_z^2}{w^2}\cos\frac{wt}{2}$, $\frac{\partial q_3}{\partial w_z} = -\frac{t}{2}\frac{w_z}{w}\sin\frac{wt}{2}$.

$$E^2(\mathbf{a}) = \sum_{\mathrm{t=s}}^{\mathrm{s+N}} \parallel \mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a}) \parallel^2, \tag{17}$$

Then the first partial derivatives of the function E^2 with respect to $\mathbf{a}_{\mathrm{i}}$ are given by:

$$\frac{\partial E^2}{\partial \mathbf{a}_{\mathrm{i}}} = 2 \sum_{t=s}^{s+N} (\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a}))^{\top} \frac{\partial}{\partial \mathbf{a}_{\mathrm{i}}} (\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})), \quad \mathrm{i} = 1, 2, \cdots, 6. \tag{18}$$

The second partial derivatives are obtained by ignoring the second derivatives of the model function:

$$\frac{\partial^2 E^2}{\partial \mathbf{a}_{\mathrm{i}} \partial \mathbf{a}_{\mathrm{j}}} \approx 2 \sum_{t=s}^{s+N} \frac{\partial}{\partial \mathbf{a}_{\mathrm{i}}} (\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a}))^{\top} \frac{\partial}{\partial \mathbf{a}_{\mathrm{j}}} (\mathbf{X}_{\mathrm{t}} - \mathbf{h}_{\mathrm{t}}(\mathbf{a})), \quad \mathrm{i,j} = 1, 2, \cdots, 6. \tag{19}$$

Let $\beta_i \stackrel{\mathrm{def}}{=} -\frac{1}{2}\frac{\partial E^2}{\partial a_i}$, and $\alpha_{ij} \stackrel{\mathrm{def}}{=} \frac{1}{2}\frac{\partial E^2}{\partial a_i \partial a_j}$, the elements in matrix $[\alpha]$. The minimization problem is reduced to iteratively solving the following linear equation:

$$\sum_{l=1}^{m} \alpha_{kl} \delta a_l = \beta_k, \tag{20}$$

where m is the number of unknown parameters. Here $m = 6$.

<u>Algorithm: TrajFit</u>

1. Compute $E^2(\mathbf{a})$ in Equation (17). Set $\lambda = 0.001$.

2. Compute β_i and α_{ij}, where $i, j = 1, 2, \cdots, 6$, using equations (18) and (19), respectively.

3. Compute matrix $[\alpha]$ by augmenting its diagonal elements: $\alpha'_{jj} = \alpha_{jj}(1 + \lambda)$, and $\alpha'_{jk} = \alpha_{jk}$.

4. Solve Equation (20) for $\delta(\mathbf{a})$ and evaluate $E^2(\mathbf{a} + \delta\mathbf{a})$.

5. If $E^2(\mathbf{a} + \delta\mathbf{a}) \geq \mathrm{E}^2(\mathbf{a})$, increase λ by a factor and go back to 2.

6. If $E^2(\mathbf{a} + \delta\mathbf{a}) < E^2(\mathbf{a})$, decrease λ by a factor, update the trial solution $\mathbf{a} \leftarrow \mathbf{a} + \delta\mathbf{a}$, and go back to 2.