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Prove that convolving an image twice with a Gaussian kernel is equivalent to convolving the image with a single Gaussian kernel with a larger standard deviation.

PROOF:

We want to prove, $I(x,y) * G_1(x,y) * G_2(x,y) = I(x,y) * G(x,y)$

where G_1 has standard deviation σ_1, G_2 has standard deviation σ_2

and G has standard deviation σ, $(\sigma > \sigma_1, \sigma_2)$.

We know,

$$\left(I(x,y) * G_1(x,y)\right) * G_2(x,y) = I(x,y) * \left(G_1(x,y) * G_2(x,y)\right)$$

$$\therefore \quad G(x,y) = G_1(x,y) * G_2(x,y)$$

It is sufficient to prove the result for the 1D case. The result for the 2D case is then, a straightforward extension.

Therefore, we have,

$$G(x) = G_1(x) * G_2(x)$$

$$G_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-x^2/2\sigma_1^2}$$

$$G_2(x) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-x^2/2\sigma_2^2}$$

$$G = G_1 * G_2 = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} e^{-k^2/2\sigma_1^2} e^{-(x-k)^2/2\sigma_2^2} \, dk$$

$$\boxed{\text{By definition, } f(x) * g(x) = \int_{-\infty}^{\infty} f(k)\, g(x-k)\, dk}$$

contd....

Therefore,

$$G = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\left(\frac{k^2}{2\sigma_1^2} + \frac{(x-k)^2}{2\sigma_2^2}\right)} dk$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{k^2}{\sigma_1^2} + \frac{x^2-2kx+k^2}{\sigma_2^2}\right)} dk$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{\sigma_2^2k^2 + \sigma_1^2x^2 - 2\sigma_1^2kx + \sigma_1^2k^2}{\sigma_1^2\sigma_2^2}\right)} dk.$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{(\sigma_1^2+\sigma_2^2)k^2 - 2\sigma_1^2kx + \sigma_1^2x^2}{\sigma_1^2\sigma_2^2}\right)} dk.$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{(\sigma_1^2+\sigma_2^2)k^2 - 2\sigma_1^2kx}{\sigma_1^2\sigma_2^2}\right]} . e^{-\frac{1}{2}\frac{\cancel{\sigma_1^2}x^2}{\cancel{\sigma_1^2}\sigma_2^2}} dk.$$

The second term is independent of 'k' and can therefore be taken outside the integral.

$$\therefore\ G = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{x^2}{2\sigma_2^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{(\sigma_1^2+\sigma_2^2)k^2}{\sigma_1^2\sigma_2^2} - \frac{2xk}{\sigma_2^2}\right]} dk.$$

Let $a = \dfrac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}$ and $b = \dfrac{x}{\sigma_2^2}$

$$\Rightarrow\ G = \frac{1}{2\pi\sigma_1\sigma_2} e^{\frac{-x^2}{2\sigma_2^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}ak^2 + bk} dk.$$

We will now make use of the identity,

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2+bx}\,dx = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}} \quad , \quad a>0} \qquad Ⓐ$$

The proof for this identity is given on page ④.

Therefore,

$$G = \left(\frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{x^2}{2\sigma_2^2}}\right)\left(\sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}\right).$$

Substituting back for 'a' and 'b', we get,

$$G = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{x^2}{2\sigma_2^2}} \sqrt{\frac{2\pi}{\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)}}\; e^{\left[\left(\frac{x}{\sigma_2^2}\right)^2 / 2\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)\right]}$$

$$= \frac{1}{2\pi\cancel{\sigma_1}\cancel{\sigma_2}} \sqrt{\frac{2\pi\sigma_1^{\cancel{2}}\sigma_2^{\cancel{2}}}{\sigma_1^2+\sigma_2^2}}\; e^{\left(\frac{\frac{x^2}{\sigma_2^{\cancel{4}}}\sigma_1^2\sigma_2^2}{2(\sigma_1^2+\sigma_2^2)}\right)} e^{-\frac{x^2}{2\sigma_2^2}}$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{\left(\frac{x^2\sigma_1^2}{2\sigma_2^2(\sigma_1^2+\sigma_2^2)} - \frac{x^2}{2\sigma_2^2}\right)}.$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{\frac{x^2}{2\sigma_2^2}\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} - 1\right)}.$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{-\frac{x^2}{2(\sigma_1^2+\sigma_2^2)}}$$

This is a Gaussian function with $\sigma_G^2 = \sigma_1^2+\sigma_2^2$, (i.e) $\sigma_G = \sqrt{\sigma_1^2+\sigma_2^2}$

If $\sigma_1 = \sigma_2 = \sigma$, then $\underline{\underline{\sigma_G = \sqrt{2}\sigma}}$

Hence proved.